EENG 479 : Digital Signal Processing (DSP)

Lecture #10: IIR Filter Design 1

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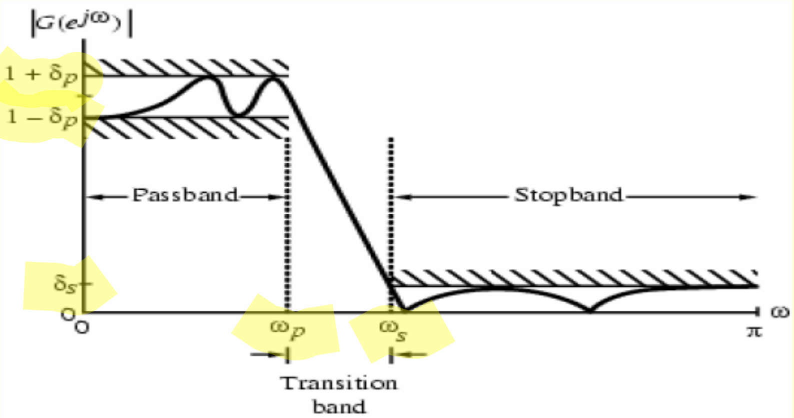
Chapter 9

Digital Filter Design

- <u>Objective</u> - Determination of a realizable transfer function $G(z)$ approximating a given frequency response specification is an important step in the development of a digital filter
- If an IIR filter is desired, $G(z)$ should be a stable real rational function
- Digital filter design is the process of deriving the transfer function $G(z)$

Digital Filter Specifications

- For example, the magnitude response $|G(e^{j\omega})|$ of a digital lowpass filter may be given as indicated below



- As indicated in the figure, in the **passband**, defined by $0 \leq \omega \leq \omega_p$, we require that $|G(e^{j\omega})| \cong 1$ with an error $\pm\delta_p$, i.e.,

$$1-\delta_p \leq |G(e^{j\omega})| \leq 1+\delta_p, \quad |\omega| \leq \omega_p$$

- In the **stopband**, defined by $\omega_s \leq \omega \leq \pi$, we require that $|G(e^{j\omega})| \cong 0$ with an error δ_s, i.e.,

$$|G(e^{j\omega})| \leq \delta_s, \quad \omega_s \leq |\omega| \leq \pi$$

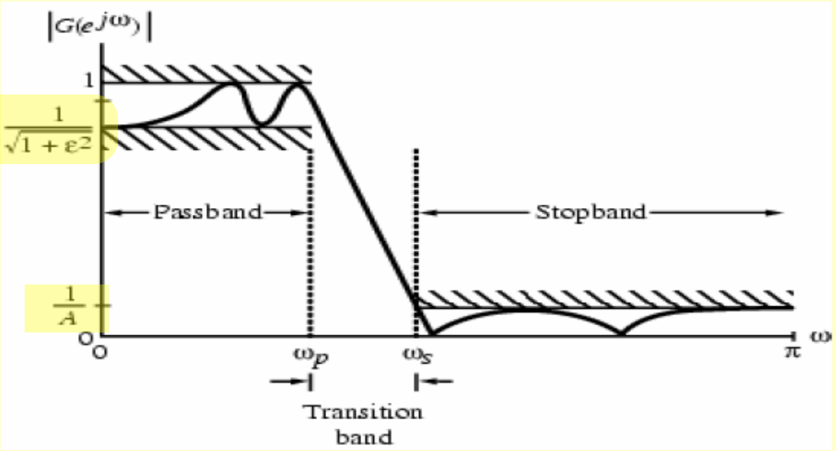
- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- Since $G(e^{j\omega})$ is a periodic function of ω, and $|G(e^{j\omega})|$ of a real-coefficient digital filter is an even function of ω
- As a result, filter specifications are given only for the frequency range $0 \le \omega \le \pi$
- Specifications are often given in terms of **loss function** $\mathcal{A}(\omega) = -20\log_{10}|G(e^{j\omega})|$ in dB
- **Peak passband ripple**

$$\alpha_p = -20\log_{10}(1-\delta_p) \quad \text{dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s) \text{ dB}$$

- Magnitude specifications may alternately be given in a normalized form as indicated below



- Here, the maximum value of the magnitude in the passband is assumed to be unity

- $1/\sqrt{1+\varepsilon^2}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband

- $\frac{1}{A}$ - Maximum stopband magnitude

- For the normalized specification, maximum value of the gain function or the minimum value of the loss function is 0 dB
- **Maximum passband attenuation** -

$$\alpha_{max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right) \text{ dB}$$

- For $\delta_p << 1$, it can be shown that

$$\alpha_{max} \cong -20\log_{10}(1-2\delta_p) \text{dB}$$

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

$$1 - \frac{1}{\sqrt{1+\varepsilon^2}} = 1+\delta_p - (1-\delta_p) = 2\delta_p$$

$$\therefore \boxed{1 - 2\delta_p = \frac{1}{\sqrt{1+\varepsilon^2}}}$$

$$\log_{10}(1-2\delta_p) = \log_{10}\left(\frac{1}{\sqrt{1+\varepsilon^2}}\right)$$

$$-\log(1-2\delta_p) = \log_{10}\left(\sqrt{1+\varepsilon^2}\right) = \alpha_{max}$$

α_{max} = max. passband attenuation

Digital Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then

$$\omega_p = \frac{2\pi(7\times10^3)}{25\times10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times10^3)}{25\times10^3} = 0.24\pi$$

Selection of Filter Type

- The transfer function $H(z)$ meeting the frequency response specifications should be a causal transfer function
- For IIR digital filter design, the IIR transfer function is a real rational function of z^{-1}:

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}, \quad M \leq N$$

- $H(z)$ must be a stable transfer function and must be of lowest order N for reduced computational complexity

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- For FIR digital filter design, the FIR transfer function is a polynomial in z^{-1} with real coefficients:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree N of $H(z)$ must be as small as possible
- If a linear phase is desired, the filter coefficients must satisfy the constraint:

$$h[n] = \pm h[N-n]$$

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase,

  (2) Filter structure always stable with quantized coefficients

- Disadvantages in using an FIR filter - Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

Digital Filter Design: Basic Approaches

- Most common approach to IIR filter design - (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications
- (2) Determine the analog lowpass filter transfer function $H_a(s)$
- (3) Transform $H_a(s)$ into the desired digital transfer function $G(z)$

- This approach has been widely used for the following reasons:

  (1) Analog approximation techniques are highly advanced

  (2) They usually yield closed-form solutions

  (3) Extensive tables are available for analog filter design

  (4) Many applications require digital simulation of analog systems

- An analog transfer function to be denoted as

$$H_a(s) = \frac{P_a(s)}{D_a(s)}$$

where the subscript "*a*" specifically indicates the analog domain

- A digital transfer function derived from $H_a(s)$ shall be denoted as

$$G(z) = \frac{P(z)}{D(z)}$$

- Basic idea behind the conversion of $H_a(s)$ into $G(z)$ is to apply a mapping from the s-domain to the z-domain so that essential properties of the analog frequency response are preserved

- Thus mapping function should be such that
  - Imaginary $(j\Omega)$ axis in the s-plane be mapped onto the unit circle of the z-plane
  - A stable analog transfer function be mapped into a stable digital transfer function

Digital Filter Design: Basic Approaches

- Three commonly used approaches to FIR filter design -

  (1) Windowed Fourier series approach

  (2) Frequency sampling approach

  (3) Computer-based optimization methods

IIR filter design from analog:

a) impulse-invariant

b) bilinear transform approach,

c) spectral transformations

Lecture 15

Bilinear Transformation Method of IIR Filter Design

IIR Digital Filter Design: Bilinear Transformation Method

- Bilinear transformation -

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

- Above transformation maps a single point in the s-plane to a unique point in the z-plane and vice-versa
- Relation between $G(z)$ and $H_a(s)$ is then given by

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

- Digital filter design consists of 3 steps:

  (1) Develop the specifications of $H_a(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$

  (2) Design $H_a(s)$

  (3) Determine $G(z)$ by applying bilinear transformation to $H_a(s)$

- As a result, the parameter T has no effect on $G(z)$ and $T = 2$ is chosen for convenience

Simplified Bilinear Transform

- Inverse bilinear transformation for $T = 2$ is

$$z = \frac{1+s}{1-s}$$

- For $s = \sigma_o + j\Omega_o$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \quad \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

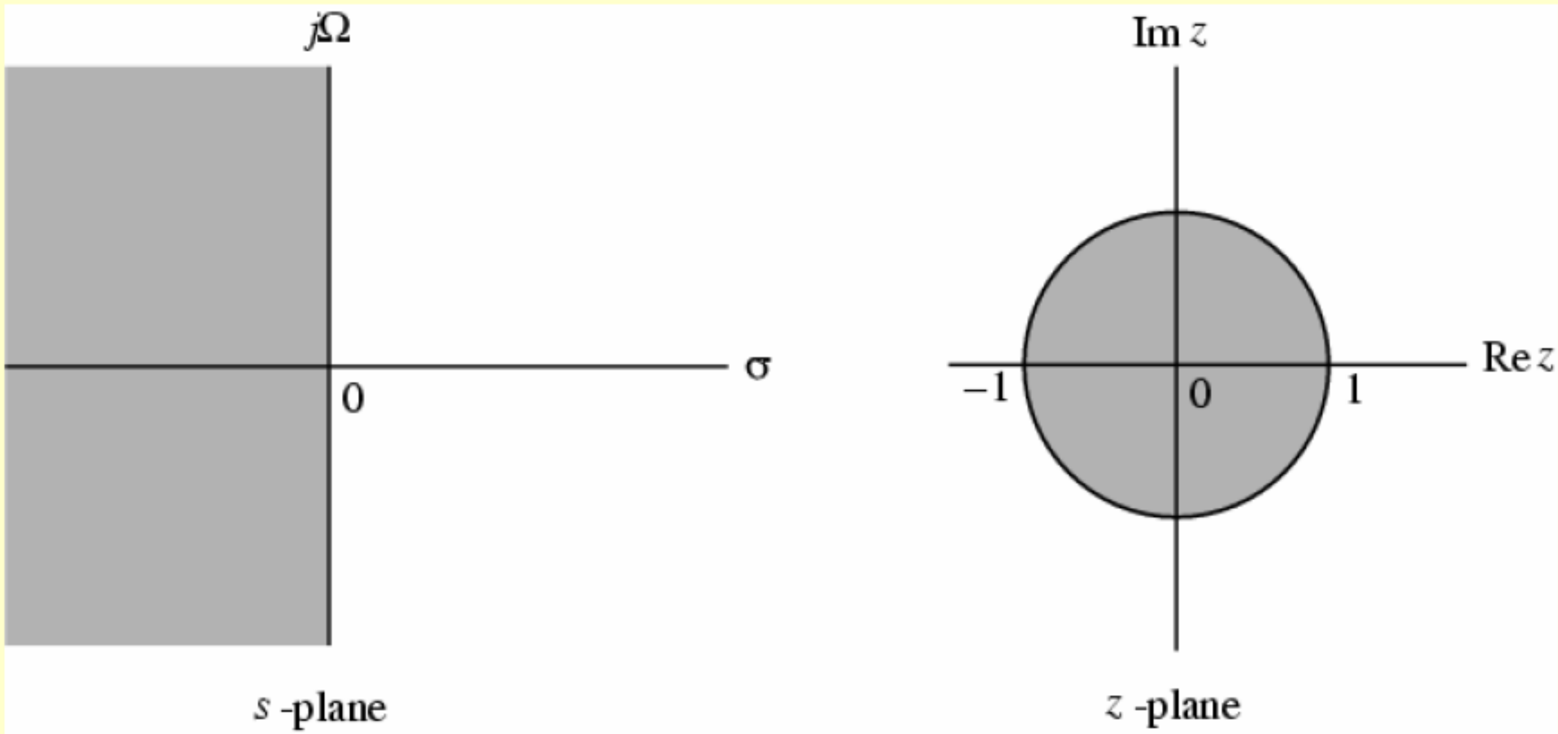
- Thus,

$$\sigma_o = 0 \rightarrow |z| = 1$$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

- Mapping of *s*-plane into the *z*-plane

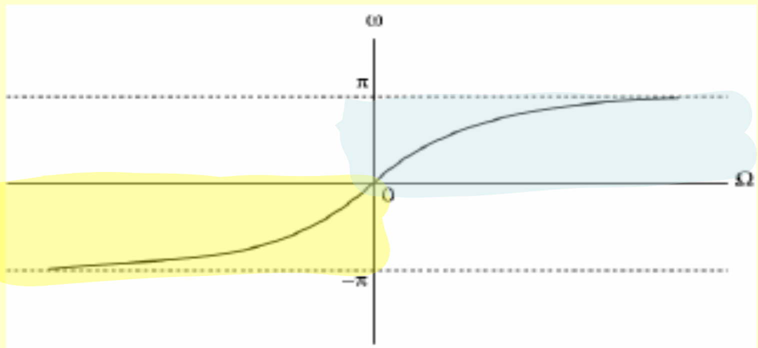


- For $z = e^{j\omega}$ with $T = 2$ we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})}$$

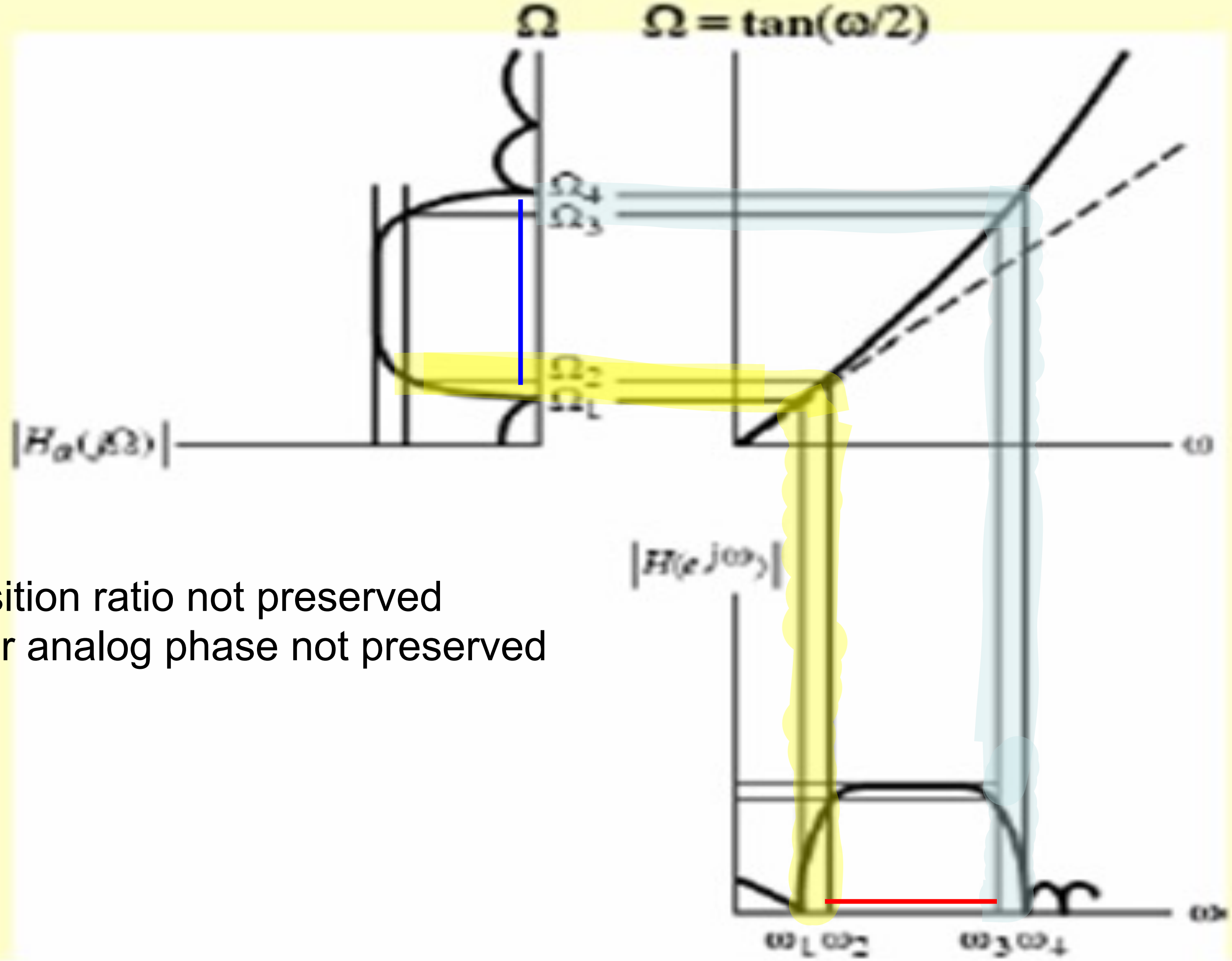
$$= \frac{j2\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2)$$

or $\Omega = \tan(\omega/2)$



- Mapping is highly nonlinear
- Complete negative imaginary axis in the *s*-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle in the *z*-plane from $z = -1$ to $z = 1$
- Complete positive imaginary axis in the *s*-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the *z*-plane from $z = 1$ to $z = -1$

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- Effect of warping shown below



Transition ratio not preserved
Linear analog phase not preserved

- Steps in the design of a digital filter -

(1) Prewarp (ω_p, ω_s) to find their analog equivalents (Ω_p, Ω_s)

(2) Design the analog filter $H_a(s)$

(3) Design the digital filter $G(z)$ by applying bilinear transformation to $H_a(s)$

- Transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values
- Transformation does not preserve phase response of analog filter

IIR Digital Filter Design Using Bilinear Transformation

- Example - Consider First order Butterworth LP digital filter

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

- Applying bilinear transformation to the above we get the transfer function of a first-order digital lowpass Butterworth filter

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1}) + \Omega_c(1+z^{-1})}$$

- Rearranging terms we get

$$G(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}}$$

where

$$\alpha = \frac{1-\Omega_c}{1+\Omega_c} = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}$$

$$G_{Lp}(z) = \frac{\Omega_c}{s+\Omega_c}\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \Rightarrow \frac{\frac{\Omega T}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{\Omega_c T}{2}(1+z^{-1})}$$

$$\frac{X(1+z^{-1})}{X-1\left(1 + \frac{X+1}{X-1} z^{-1}\right)}$$

$\frac{X}{X-1} \rightarrow \frac{1-\alpha}{2}$ $\qquad$ $\frac{X+1}{X-1} \rightarrow \alpha$

$$G_{Lp}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

$$\alpha = \frac{1-\frac{\Omega_c T}{2}}{1+\frac{\Omega_c T}{2}}$$

See book examples

- First order Butterworth HP digital filter
- Second order Bandpass digital filter
- Second order Bandstop digital filter (next)

- Example - Consider the second-order analog notch transfer function

$$H_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + Bs + \Omega_o^2}$$

for which $|H_a(j\Omega_o)| = 0$

$$|H_a(j0)| = |H_a(j\infty)| = 1$$

- Ω_o is called the **notch frequency**
- If $|H_a(j\Omega_2)| = |H_a(j\Omega_1)| = 1/\sqrt{2}$ then $B = \Omega_2 - \Omega_1$ is the **3-dB notch bandwidth**
- Then $G(z) = H_a(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

$$= \frac{(1+\Omega_o^2) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2)z^{-2}}{(1+\Omega_o^2+B) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2-B)z^{-2}}$$

$$= \frac{1+\alpha}{2} \cdot \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - 2\beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

where $\alpha = \frac{1+\Omega_o^2 - B}{1+\Omega_o^2 + B} = \frac{1-\tan(B_w/2)}{1+\tan(B_w/2)}$

$$\beta = \frac{1-\Omega_o^2}{1+\Omega_o^2} = \cos\omega_o$$

- Example - Design a 2nd-order digital notch filter operating at a sampling rate of 400 Hz with a notch frequency at 60 Hz, 3-dB notch bandwidth of 6 Hz
- Thus $\omega_o = 2\pi(60/400) = 0.3\pi$

$$B_w = 2\pi(6/400) = 0.03\pi$$

- From the above values we get

$$\alpha = 0.90993$$

$$\beta = 0.587785$$

- Thus

$$G(z) = \frac{0.954965 - 1.1226287\,z^{-1} + 0.954965\,z^{-2}}{1 - 1.1226287\,z^{-1} + 0.90993\,z^{-2}}$$

- The gain and phase responses are shown below