EENG 479 : Digital Signal Processing (DSP)

Lecture #11: IIR Filter Design 2

Prof. Mohab A. Mangoud
Professor of Wireless Communications (Networks, IoT and AI)
University of Bahrain, College of Engineering
Department of Electrical and Electronics Engineering
P.O.Box 32038- Kingdom of Bahrain
mmangoud@uob.edu.bh
http://mangoud.com

IIR Lowpass Digital Filter Design Using Bilinear Transformation

- Example - Design a lowpass Butterworth digital filter with $\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$, $\alpha_p \leq 0.5\,\text{dB}$, and $\alpha_s \geq 15\ \ \text{dB}$

```
>> (10^(15/20))^2

ans =

   31.6228
```

- Minimum stopband attenuation $\alpha_s = -20\log_{10}(\delta_s)\ \text{dB}$

- Thus

$$\varepsilon^2 = 0.1220185 \qquad A^2 = 31.622777$$

Maximum passband attenuation - $\alpha_{max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right)\ \text{dB}$

- If $\left|G(e^{j0})\right| = 1$ this implies

$$20\log_{10}\left|G(e^{j0.25\pi})\right| \geq -0.5$$

$$20\log_{10}\left|G(e^{j0.55\pi})\right| \leq -15$$

```
>> 10^(0.5/20)

ans =

    1.0593

>> ans^2

ans =

    1.1220
```

- Prewarping we get

$$\Omega_p = \tan(\omega_p/2) = \tan(0.25\pi/2) = 0.4142136$$

$$\Omega_s = \tan(\omega_s/2) = \tan(0.55\pi/2) = 1.1708496$$

- The inverse transition ratio is

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.8266809$$

- The inverse discrimination ratio is

$$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 15.841979$$

- Thus $N = \dfrac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 2.6586997$

[from Butterworth_Analog review]

- We choose $N = 3$
- To determine Ω_c we use

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

- We then get

$$\Omega_c = 1.419915(\Omega_p) = 0.588148$$

- 3rd-order lowpass Butterworth transfer function for $\Omega_c = 1$ is

(Matlab)

$$\boxed{H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}}$$

- Denormalizing to get $\Omega_c = 0.588148$ we arrive at

$$H_a(s) = H_{an}\left(\frac{s}{0.588148}\right)$$

- BUTTAP :Butterworth analog lowpass filter prototype.

[Z,P,K] = BUTTAP(N) returns the zeros, poles, and gain for an N-th order normalized prototype Butterworth analog lowpass filter. The resulting filter has N poles around the unit circle in the left half plane, and no zeros.

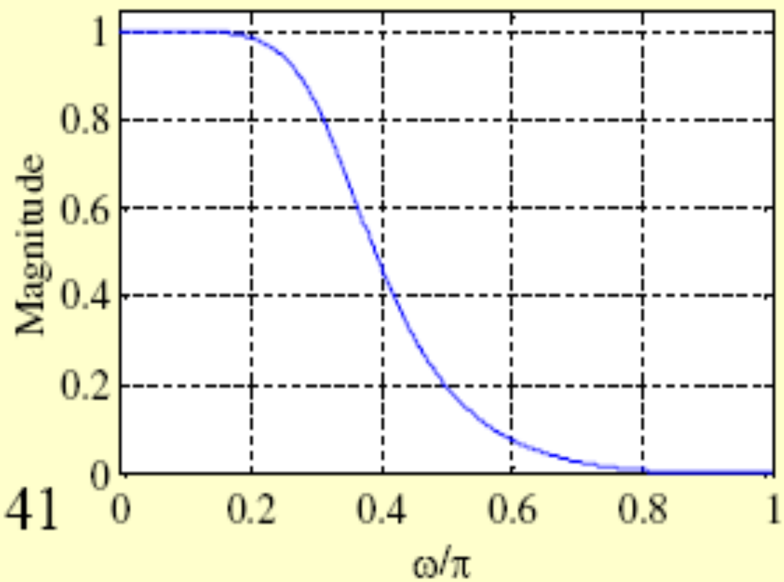
Example:

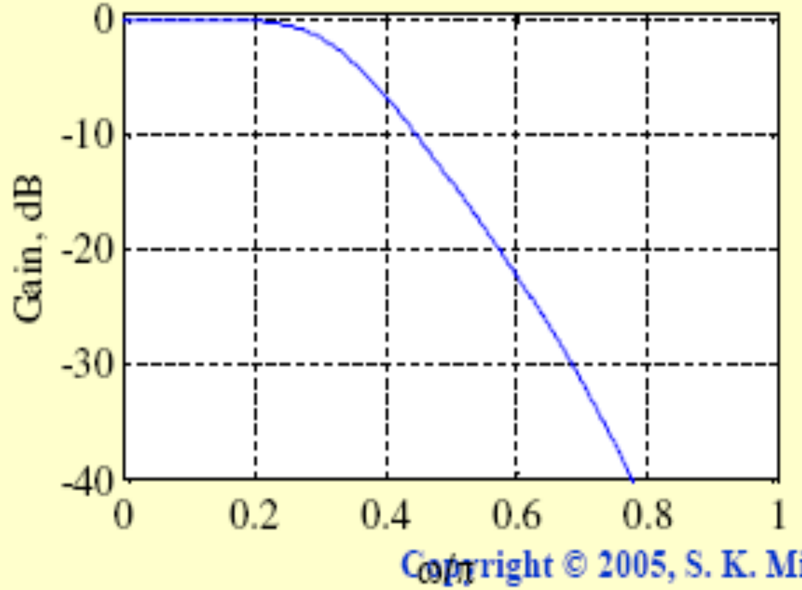
```
>> [Z,P,K] = BUTTAP(3)
Z =     []
P =  -0.5000 + 0.8660i,   -0.5000 - 0.8660i,  -1.0000
K =    1.0000

>>TF2SOS(den,1)

ans =

  1.00    1.00    0       1.00    0    0
  1.00    1.00    1.00    1.00    0    0
```

- Applying bilinear transformation to $H_a(s)$ we get the desired digital transfer function

$$\boxed{G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}}$$

- Magnitude and gain responses of $G(z)$ shown below:

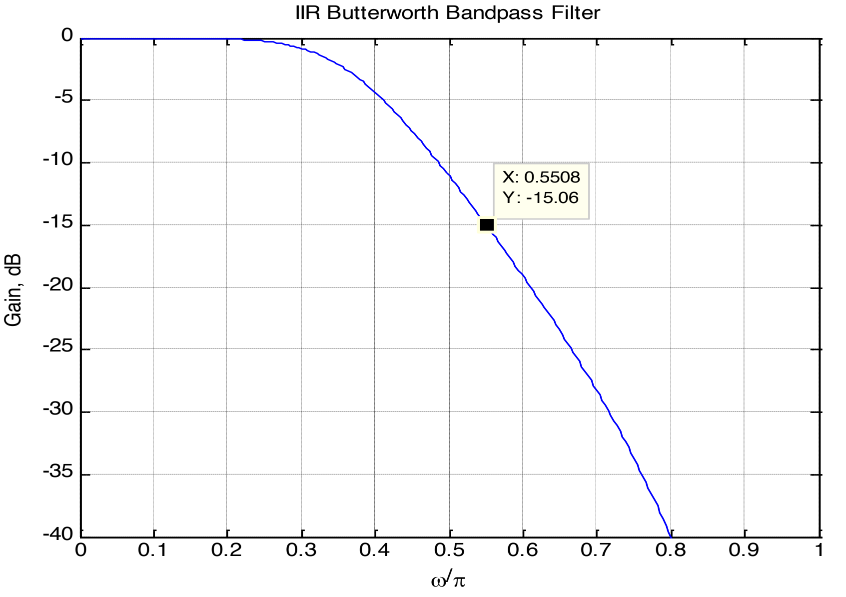




Note: The above digital LP filter can be designed directly in the z domain using Buttord and butter commands

```
% Program 9_3
% Design of IIR Butterworth Bandpass Filter
Wp = input('Passband edge frequencies = ');
Ws = input('Stopband edge frequencies = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Minimum stopband attenuation = ');
[N,Wn] = buttord(Wp, Ws, Rp, Rs);
[b,a] = butter(N,Wn);
[h,omega] = freqz(b,a,256);
gain = 20*log10(abs(h));
plot (omega/pi,gain);grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('IIR Butterworth Bandpass Filter');
```

```
>> Program_9_3
Passband edge frequencies = 0.25
Stopband edge frequencies = 0.55
Passband ripple in dB = -0.5
Minimum stopband attenuation = -15
```



FREQZ Digital filter frequency response.

[H,W] = FREQZ(B,A,N) returns the N-point complex frequency response vector H and the N-point frequency vector W in radians/sample of the filter:

```
             jw               -jw              -jmw
         jw  B(e)    b(1) + b(2)e + .... + b(m+1)e
     H(e) = ---- = ------------------------------------
             jw               -jw              -jnw
             A(e)    a(1) + a(2)e + .... + a(n+1)e
```

given numerator and denominator coefficients in vectors B and A. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

BUTTORD Butterworth filter order selection.

[N, Wn] = BUTTORD(Wp, Ws, Rp, Rs) returns the order N of the lowest order digital Butterworth filter that loses no more than Rp dB in the passband and has at least Rs dB of attenuation in the stopband. Wp and Ws are the passband and stopband edge frequencies, normalized from 0 to 1 (where 1 corresponds to pi radians/sample). For example,

Lowpass: Wp = .1, Ws = .2
Highpass: Wp = .2, Ws = .1
Bandpass: Wp = [.2 .7], Ws = [.1 .8]
Bandstop: Wp = [.1 .8], Ws = [.2 .7]

BUTTORD also returns Wn, the Butterworth natural frequency (or, the "3 dB frequency") to use with BUTTER to achieve the specifications.

[N, Wn] = BUTTORD(Wp, Ws, Rp, Rs, 's') does the computation for an analog filter, in which case Wp and Ws are in radians/second.

When Rp is chosen as 3 dB, the Wn in BUTTER is equal to Wp in BUTTORD.

BUTTER Butterworth digital and analog filter design.

[B,A] = BUTTER(N,Wn) designs an Nth order lowpass digital

Butterworth filter and returns the filter coefficients in length N+1 vectors B (numerator) and A (denominator).

Extra material from A.A.Beex lecture notes

- **Parametric Bilinear Transform**
- **Numerical example**

Bilinear Transform method

(from A.A.Beex lecture notes)

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \longrightarrow s = \frac{2}{T}\frac{z-1}{z+1} \Rightarrow \frac{sT}{2}(z+1) = z-1$$

$$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

$s=\sigma+j\Omega$

$$z = \frac{1+\frac{\sigma T}{2}+j\frac{\Omega T}{2}}{1-\frac{\sigma T}{2}-j\frac{\Omega T}{2}}$$

$\sigma>0$

$|z|>1$

$\sigma<0 \Leftrightarrow$ BIBO analog

$$|z| = \frac{|1-\varepsilon+jx|}{|1+\varepsilon-jx|} < 1 \Leftrightarrow \text{BIBO digital}$$

$s=j\Omega$

$$z = \frac{1+j\Omega T/2}{1-j\Omega T/2}$$

$$|z| = \frac{|1+jx|}{|1-jx|} = 1$$

$s=j\Omega \Longrightarrow |z|=1$

$j\Omega$ maps to u.c.

$z=e^{j\omega}$

$$s = \frac{2}{T}\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2}{T}\frac{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2}+e^{-j\omega/2})} = \frac{2}{T}\frac{2j\sin\omega/2}{2\cos\omega/2} = \frac{2}{T}j\tan(\omega/2)$$

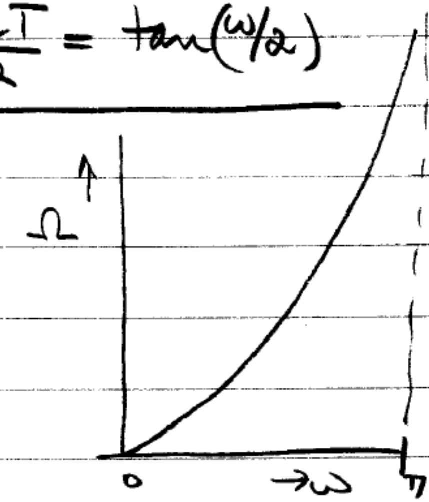
$$= j\Omega$$

$$= j\Omega$$

$$\Downarrow$$

$$\frac{\Omega T}{2} = \tan(\omega/2)$$

$$\boxed{G(z) = H_a(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}}$$



$$H(s) = C\,\frac{\prod_{m=1}^{M}(s-\sigma_m)}{\prod_{k=1}^{N}(s-p_k)} \xrightarrow{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} G(z) \qquad **$$

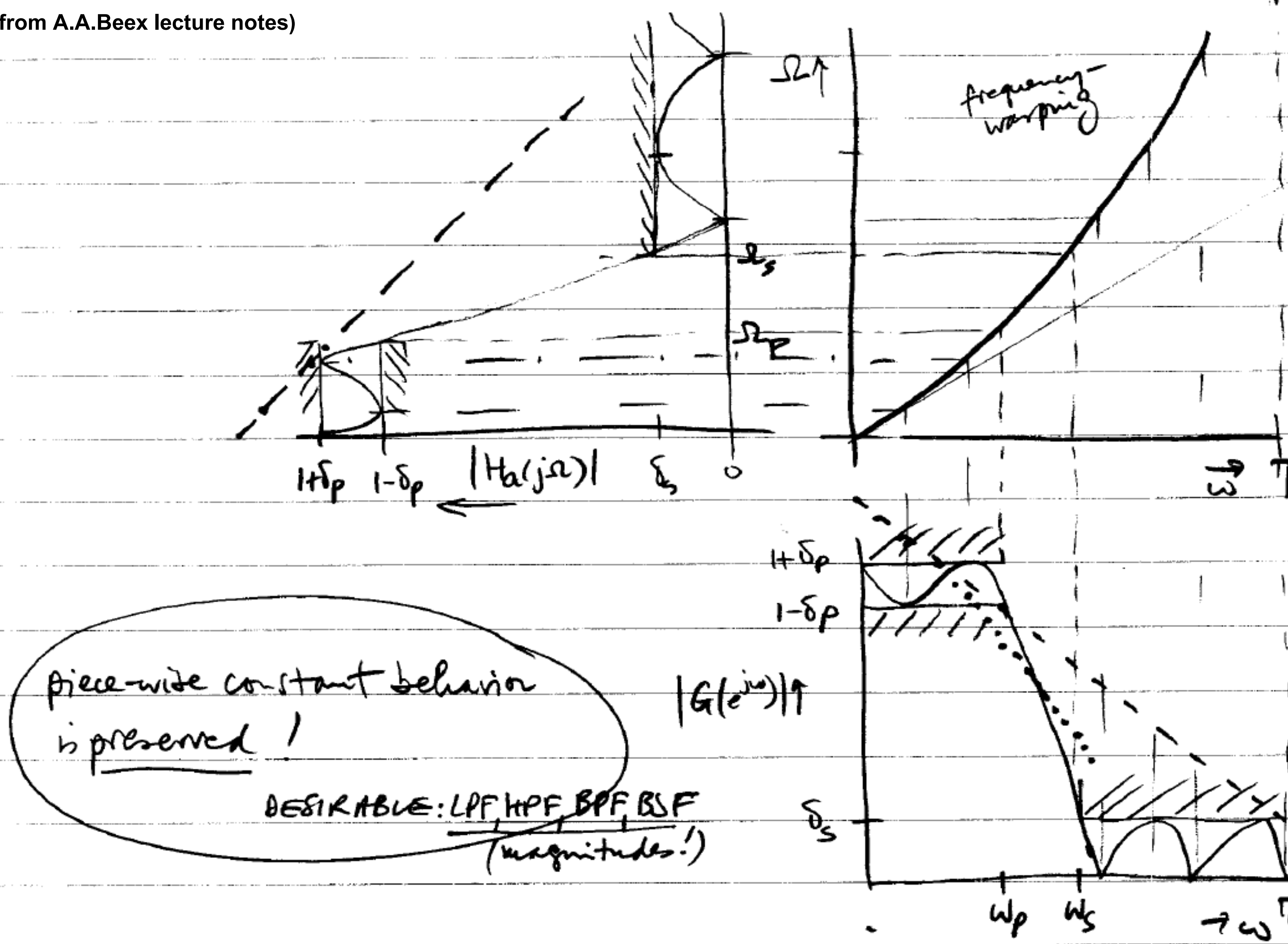
$$\Downarrow$$

$$s = j\Omega \longrightarrow z = e^{j\omega}$$

$$\left.\begin{aligned} |H(s)| &= |G(z)| \\ \angle H(s) &= \angle G(z) \end{aligned}\right\} \text{ for } \frac{\Omega T}{2} = \tan(\omega/2)$$

(from A.A.Beex lecture notes)

(from A.A.Beex lecture notes)



(*) linear analog phase is not preserved!

$\Omega_s/\Omega_p \neq \omega_s/\omega_p$ transition ratios not preserved!

BT: parametric transformation (imagine doing it for high order poly!)

$$s-p_k = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} - p_k = \frac{2(1-z^{-1}) - p_k T(1+z^{-1})}{T(1+z^{-1})}$$

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$= \frac{(2-p_k T) - (2+p_k T)z^{-1}}{T(1+z^{-1})}$$

analog pole $\xrightarrow{\text{digital pole}}$ $\xrightarrow{\text{digital zero}}$

$$\boxed{= \frac{(2-p_k T)\left(1 - \frac{2+p_k T}{2-p_k T} z^{-1}\right)}{T(1+z^{-1})}}$$

(from A.A.Beex lecture notes)

$$\Rightarrow G_{BT}(z) = \underbrace{C \cdot T^{N-M} \frac{\prod_{m=1}^{M}(2-\sigma_m T)}{\prod_{k=1}^{N}(2-p_k T)}}_{\text{gain constant}} \cdot (1+z^{-1})^{N-M} \underbrace{\frac{\prod_{m=1}^{M}\left(1-\frac{2+\sigma_m T}{2-\sigma_m T}z^{-1}\right)}{\prod_{k=1}^{N}\left(1-\frac{2+p_k T}{2-p_k T}z^{-1}\right)}}_{\text{combine these into SOS (real)}}$$

from zeros of $H_a(s)$ @ $s=\infty$ (mapped to $z=-1$)

[BW & Ch I are all pole
⇓
N digital zeros at -1]

(from A.A.Beex lecture notes)

Example

Desired discrete-time elliptic lowpass

$$\left.\begin{array}{l} \dfrac{\omega_p}{2\pi} = 1000\ Hz \longrightarrow \omega_p = 0.2\pi \\[2ex] \dfrac{\omega_s}{2\pi} = 1{,}500\ Hz \longrightarrow \omega_s = 0.3\pi \end{array}\right\} \omega_s/\omega_p = 1.5$$

$$\Uparrow$$

digital specs $\quad f_s = 10\ kHz \rightarrow T = 10^{-4}\ s$

$$1-\delta_p = -0.3\ dB$$

$$\delta_s = -50\ dB$$

① **prewarp** for $T=1$

$$\frac{\Omega T}{2} = \tan(\omega/2) \Rightarrow \begin{array}{l} \Omega_p = 0.6498 \text{ rad/s} \\ \Omega_s = 1.01905 \text{ rad/s} \end{array} \Bigg\} \; \Omega_s/\Omega_p = 1.568$$

a little less strict than digital

② **Tables** (software)

$$\left.\begin{array}{l} 1-\delta_p = -0.28 \text{ dB} \\ \boxed{N=5} \\ \Omega_s/\Omega_p = 1.556 \\ \delta_s = -50.10 \text{ dB} \end{array}\right\}$$

Standardized tables

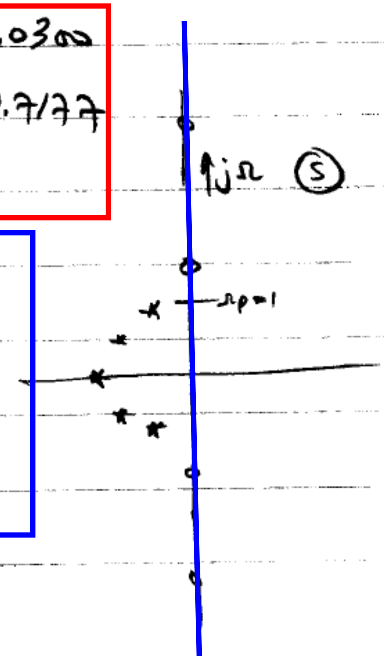
$\underline{\Omega_p \equiv 1 \text{ rad/s}}$

$\Downarrow$

poles $\begin{cases} p_{1,2} = -0.09699 \pm j\,1.0300 \\ p_{3,4} = -0.33390 \pm j\,0.7177 \\ p_5 = -0.49519 \end{cases}$

zeros $\begin{cases} \sigma_{1,2} = \pm j\,1.6170 \\ \sigma_{3,4} = \pm j\,2.4377 \\ \sigma_5 = \infty \end{cases}$

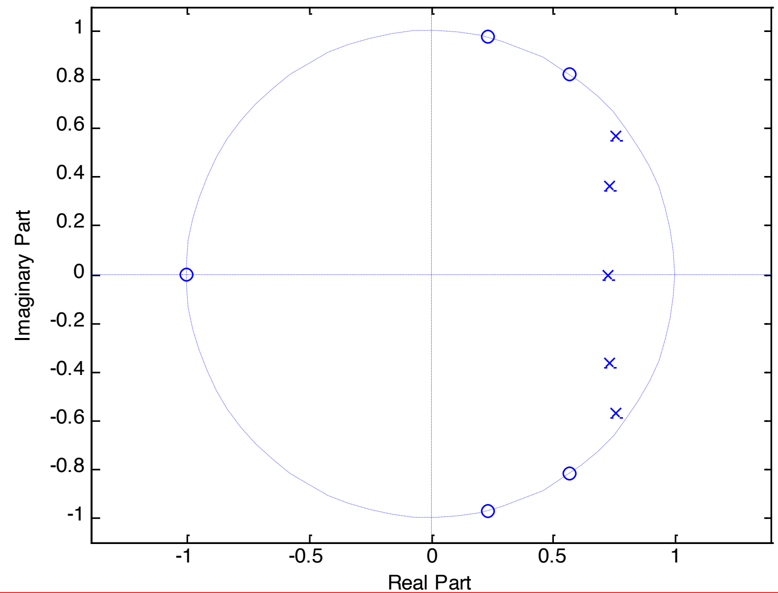
not in table




```
>> [N, Wn] = ELLIPORD(0.6498, 1.101905, 0.3, 50, 's')

N =

     5


Wn =

    0.6498
```

(from A.A.Beex lecture notes)

```
>> [B,A]=ELLIP(N,0.3,50,1,'low','s')

B =

         0    0.0214    0.0000    0.1804    0.0000    0.3226


A =

    1.0000    1.3335    2.2235    1.6999    1.0599    0.3226

>> roots(B)

ans =

  -0.0000 + 2.4170i
  -0.0000 - 2.4170i
        0 + 1.6053i
        0 - 1.6053i

>> roots(A)

ans =

  -0.0950 + 1.0278i
  -0.0950 - 1.0278i
  -0.3280 + 0.7165i
  -0.3280 - 0.7165i
  -0.4875
```



③ de-normalize analog design

$(s-\sigma_0) \sim (j\Omega - \sigma_0)$

$\Downarrow$

$(j\Omega/\Omega_p - \sigma_0)$

$\Downarrow$

$(j\Omega - \Omega_p\sigma_0) \Rightarrow$ replace σ_m by $\Omega_p\sigma_m$ & ρ_k by $\Omega_p\rho_k$

to get $\Omega_p = 0.6498$ i.s.o. $\Omega_p = 1$

(from A.A.Beex lecture notes)

(4) BT transform poles & zeros

$$(s-\Omega_p\sigma_m) \longrightarrow 1-\frac{2+T\Omega_p\sigma_m}{2-T\Omega_p\sigma_m}z^{-1} \quad \text{etc.}$$

$p_{1,2} = 0.7592 \pm j\,0.5692 = 0.9499\angle 37.04^\circ \rightarrow 1-1.5084\,z^{-1}+0.8928\,z^{-2}$

$p_{3,4} = 0.7278 \pm j\,0.3635 = 0.8135\angle 26.54^\circ \rightarrow 1-1.4556\,z^{-1}+0.6618\,z^{-2}$

$p_5 = 0.7228 \quad = \text{---} \rightarrow 1-0.7228\,z^{-1}$

$z_{1,2} = 0.5673 \pm j\,0.8235 = 1.0\angle 55.44^\circ \rightarrow 1-1.1346\,z^{-1}+z^{-2}$

$z_{3,4} = 0.2290 \pm j\,0.9734 = 1.0\angle 76.76^\circ \rightarrow 1-0.4580\,z^{-1}+z^{-2}$

$\Rightarrow z_5 = -1 \rightarrow 1+z^{-1}$

(from A.A.Beex lecture notes)

```
>> [B,A]=ELLIP(N,0.3,50,wn)

B =

    0.0082   -0.0051    0.0076    0.0076   -0.0051    0.0082


A =

    1.0000   -3.6997    5.9309   -5.0551    2.2788   -0.4334

>> roots(A)

ans =

   0.7558 + 0.5688i
   0.7558 - 0.5688i
   0.7308 + 0.3641i
   0.7308 - 0.3641i
   0.7265

>> roots(B)

ans =

  -1.0000
   0.5723 + 0.8201i
   0.5723 - 0.8201i
   0.2370 + 0.9715i
   0.2370 - 0.9715i
```

```
>> TF2SOS(B,A)

ans =

    0.0082    0.0082         0    1.0000   -0.7265         0
    1.0000   -0.4741    1.0000    1.0000   -1.4615    0.6666
    1.0000   -1.1445    1.0000    1.0000   -1.5117    0.8949
```

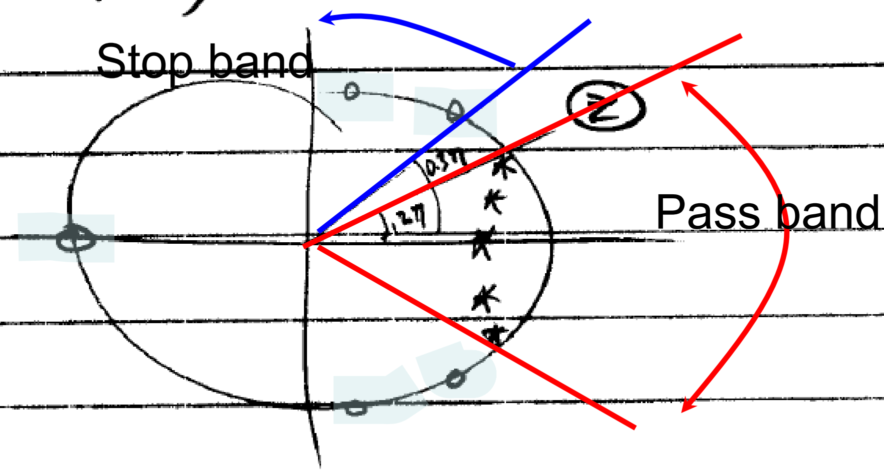
(5) Determine gain from BT trafo, or by gain-scaling

eg. $G(z=1) = 1 \Rightarrow K = 0.0082325$

$\pi \sim 5\,kHz$

$\frac{55.49^\circ}{180}\pi = 0.9673 = 0.3080\pi$

$\frac{76.76^\circ}{180^\circ}\pi = 1.3397 = 0.4264\pi$



(from A.A.Beex lecture notes)