EENG 479 : Digital Signal Processing (DSP)

Lecture #5:

2.7 Finite Dimensional LTI Discrete Time Systems

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2.7 Finite-Dimensional LTI Discrete-Time Systems

An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$\boxed{\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k],} \tag{2.90}$$

where $x[n]$ and $y[n]$ are, respectively, the input and the output of the system, and $\{d_k\}$ and $\{p_k\}$ are constants. The *order* of the discrete-time system is given by $\max(N, M)$, which is the order of the difference equation characterizing the system. It is possible to implement an LTI system characterized by Eq. (2.90) since the computation here involves two finite sums of products even though such a system, in general, has an impulse response of infinite length.

The output $y[n]$ can then be computed recursively from Eq. (2.90). If we assume the system to be causal, then we can rewrite Eq. (2.90) to express $y[n]$ explicitly as a function of $x[n]$:

$$\boxed{y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k],} \tag{2.91}$$

provided $d_0 \neq 0$. The output $y[n]$ can be computed for all $n \geq n_o$, knowing $x[n]$ and the initial conditions $y[n_o - 1], y[n_o - 2], \ldots, y[n_o - N]$.

A simple finite-dimensional LTI system is considered in Example 2.36.

2.7.1 Total Solution Calculation

The procedure for computing the solution of the constant coefficient difference equation of Eq. (2.90) is very similar to that employed in solving the constant coefficient differential equation in the case of an LTI continuous-time system. In the case of the discrete-time system of Eq. (2.90), the output response $y[n]$ also consists of two components that are computed independently and then added to yield the total solution:

$$\boxed{y[n] = y_c[n] + y_p[n].} \tag{2.97}$$

In Eq. (2.97), the component $y_c[n]$ is the solution of Eq. (2.90) with the input $x[n] = 0$; that is, it is the solution of the homogeneous difference equation:

$$\boxed{\sum_{k=0}^{N} d_k y[n-k] = 0,} \tag{2.98}$$

and the component $y_p[n]$ is a solution of Eq. (2.90) with $x[n] \neq 0$. $y_c[n]$ is called the *complementary solution* or *homogeneous solution,* while $y_p[n]$ is called the *particular solution,* resulting from the specified input $x[n]$, often called the *forcing function.* The sum of the complementary and the particular solutions as given by Eq. (2.97) is called the *total solution.*

We first describe the method of computing the complementary solution $y_c[n]$. To this end, we assume that it is of the form

$$\boxed{y_c[n] = \lambda^n.} \tag{2.99}$$

Substituting the above in Eq. (2.98), we arrive at

$$\boxed{\begin{aligned}\sum_{k=0}^{N} d_k y[n-k] &= \sum_{k=0}^{N} d_k \lambda^{n-k} \\ &= \lambda^{n-N}(d_0\lambda^N + d_1\lambda^{N-1} + \cdots + d_{N-1}\lambda + d_N) = 0.\end{aligned}} \tag{2.100}$$

The polynomial $\sum_{k=0}^{N} d_k \lambda^{N-k}$ is called the *characteristic polynomial* of the discrete-time system of Eq. (2.90). Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote its N roots. If these roots are all distinct, then the general form of the complementary solution is given by

$$\boxed{y_c[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_N \lambda_N^n,} \tag{2.101}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are constants determined from the specified initial conditions of the discrete-time system. The complementary solution takes a different form in the case of multiple roots. For example, if λ_1 is of multiplicity L and the remaining $N-L$ roots, $\lambda_2, \lambda_3, \ldots, \lambda_{N-L}$, are distinct, then Eq. (2.101) takes the form

$$\boxed{y_c[n] = \alpha_1 \lambda_1^n + \alpha_2 n \lambda_1^n + \alpha_3 n^2 \lambda_1^n + \cdots + \alpha_L n^{L-1} \lambda_1^n + \alpha_{L+1} \lambda_2^n + \cdots + \alpha_N \lambda_{N-L}^n.} \tag{2.102}$$

Next, we consider the determination of the particular solution $y_p[n]$ of the difference equation of Eq. (2.90). Here the procedure is to assume that the particular solution is also of the same form as the specified input $x[n]$ if $x[n]$ has the form λ_0^n $(\lambda_0 \neq \lambda_i, i = 1, 2, \ldots, N)$ for all n. Thus, if $x[n]$ is a constant, then $y_p[n]$ is also assumed to be constant. Likewise, if $x[n]$ is a sinusoidal sequence, then $y_p[n]$ is also assumed to be a sinusoidal sequence, and so on.

We illustrate the determination of the total solution in Example 2.37.

EXAMPLE 2.37 Total Solution Computation of an LTI System for a Constant Input

Let us determine the total solution for $n \geq 0$ of a discrete-time system characterized by the following difference equation:

$$y[n] + y[n-1] - 6y[n-2] = x[n], \tag{2.103}$$

for a step input $x[n] = 8\mu[n]$ and with initial conditions $y[-1] = 1$ and $y[-2] = -1$.

We first determine the form of the complementary solution. Setting $x[n] = 0$ and $y[n] = \lambda^n$ in Eq. (2.103), we arrive at

$$\begin{aligned} \lambda^n + \lambda^{n-1} - 6\lambda^{n-2} &= \lambda^{n-2}(\lambda^2 + \lambda - 6) \\ &= \lambda^{n-2}(\lambda + 3)(\lambda - 2) = 0, \end{aligned}$$

and hence the roots of the characteristic polynomial $\lambda^2 + \lambda - 6$ are $\lambda_1 = -3, \lambda_2 = 2$. Therefore, the complementary solution is of the form

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n. \tag{2.104}$$

For the particular solution, we assume

$$y_p[n] = \beta.$$

Substituting the above in Eq. (2.103), we get

$$\beta + \beta - 6\beta = 8\mu[n].$$

which for $n \geq 0$ yields $\beta = -2$.

The total solution is therefore of the form

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0. \tag{2.105}$$

The constants α_1 and α_2 are chosen to satisfy the specified initial conditions. From Eqs. (2.103) and (2.105), we get

$$\begin{aligned} y[-2] &= \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 2 = -1, \\ y[-1] &= \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 2 = 1. \end{aligned}$$

Solving these two equations, we arrive at

$$\alpha_1 = -1.8, \;\; \alpha_2 = 4.8.$$

Thus, the total solution is given by

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0 \tag{2.106}$$

EXAMPLE 2.38 **Total Solution Computation of an LTI System for an Exponential Input**

We determine the total solution for $n \geq 0$ of the difference equation of Eq. (2.103) for an input $\boxed{x[n] = 2^n \mu[n]}$ with the same initial conditions as in Example 2.37.

As indicated in Example 2.37, the complementary solution contains a term $\alpha_2 (2)^n$, which is of the same form as the specified input. Hence, we need to select a form for the particular solution that is distinct and does not contain any terms similar to those contained in the complementary solution. We assume

$$\boxed{y_p[n] = \beta n (2)^n.}$$

Substituting the above in Eq. (2.103), we get

$$\beta n (2)^n + \beta (n-1)(2)^{n-1} - 6\beta (n-2)(2)^{n-2} = (2)^n \mu[n].$$

For $n \geq 0$, we obtain from the above equation $\beta = 0.4$. The total solution is now of the form

$$\boxed{y[n] = \alpha_1 (-3)^n + \alpha_2 (2)^n + 0.4 n (2)^n, \qquad n \geq 0.} \tag{2.107}$$

To determine the values of α_1 and α_2, we make use of the specified initial conditions. From Eqs. (2.103) and (2.107), we arrive at

$$y[-2] = \alpha_1 (-3)^{-2} + \alpha_2 (2)^{-2} + 0.4(-2)(2)^{-2} = -1,$$
$$y[-1] = \alpha_1 (-3)^{-1} + \alpha_2 (2)^{-1} + 0.4(-1)(2)^{-1} = 1,$$

which when solved yields $\alpha_1 = -5.04$, $\alpha_2 = -0.96$. Therefore, the total solution is given by

$$\boxed{y[n] = -5.04(-3)^n - 0.96(2)^n + 0.4 n (2)^n, \qquad n \geq 0.}$$

2.7.2 Zero-Input Response and Zero-State Response

An alternate approach to determining the total solution $y[n]$ of the difference equation of Eq. (2.90) is by computing its *zero-input response*, $y_{zi}[n]$, and *zero-state response*, $y_{zs}[n]$. The component $y_{zi}[n]$ is obtained by solving Eq. (2.90) by setting the input $x[n] = 0$, and the component $y_{zs}[n]$ is obtained by solving Eq. (2.90) by applying the specified input with all initial conditions set to zero. The total solution is then given by $y_{zi}[n] + y_{zs}[n]$.

This approach is illustrated in Example 2.39.

EXAMPLE 2.39 Total Solution Computation from Zero-Input and Zero-State Responses

We determine the total solution of the discrete-time system of Example 2.37 by computing the zero-input response and the zero-state response.

$$y[n] + y[n-1] - 6y[n-2] = x[n],$$

The zero-input response, $y_{zi}[n]$, of Eq. (2.103) is given by the complementary solution of Eq. (2.104), where the constants α_1 and α_2 are chosen to satisfy the specified initial conditions. Now, from Eq. (2.103), we get

$$y[0] = -y[-1] + 6y[-2] = -1 - 6 = -7, \qquad y[1] = -y[0] + 6y[-1] = 7 + 6 = 13.$$

Next, from Eq. (2.104), we get

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n.$$

$$y[0] = \alpha_1 + \alpha_2, \qquad y[1] = -3\alpha_1 + 2\alpha_2.$$

Solving these two sets of equations, we arrive at $\alpha_1 = -5.4$, $\alpha_2 = -1.6$. Therefore,

$$y_{zi}[n] = -5.4(-3)^n - 1.6(2)^n, \qquad n \geq 0.$$

The zero-state response is determined from Eq. (2.105) by evaluating the constants α_1 and α_2 to satisfy the zero initial conditions. From Eq. (2.103), we get

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0.$$

$$y[0] = x[0] = 8, \qquad y[1] = x[1] - y[0] = 0.$$

Next, from Eq. (2.105) and the above set of equations, we arrive at $\alpha_1 = 3.6$, $\alpha_2 = 6.4$. Thus, the zero-state response for $n \geq 0$ with initial conditions $y_{zs}[-2] = y_{zs}[-1] = 0$ is given by

$$y_{zs}[n] = 3.6(-3)^n + 6.4(2)^n - 2.$$

Hence, the total solution $y[n]$ is given by the sum $y_{zi}[n] + y_{zs}[n]$, resulting in

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0,$$

which is identical to that derived in Example 2.37, as expected.

2.7.3 Impulse Response Calculation

The impulse response $h[n]$ of a causal LTI discrete-time system is the output observed with input $x[n] = \delta[n]$. Thus, it is simply the zero-state response with $x[n] = \delta[n]$. Now for such an input, $x[n] = 0$ for $n > 0$, and thus, the particular solution is zero, that is, $y_p[n] = 0$. Hence, the impulse response can be computed from the complementary solution of Eq. (2.101) in the case of simple roots of the characteristic equation by determining the constants α_i to satisfy the zero initial conditions. A similar procedure can be followed in the case of multiple roots of the characteristic equation. A system with all zero initial conditions is often called a *relaxed* system.

We illustrate the impulse response computation in Examples 2.40 and 2.41.

EXAMPLE 2.40 **Impulse Response Computation from Zero-State Response**

In this example, we determine the impulse response $h[n]$ of the causal discrete-time system of Example 2.37. From Eq. (2.104), we get

$$\boxed{h[n] = \alpha_1(-3)^n + \alpha_2(2)^n, \qquad n \geq 0.}$$

From the above, we arrive at

$$h[0] = \alpha_1 + \alpha_2, \qquad h[1] = -3\alpha_1 + 2\alpha_2.$$

Next, from Eq. (2.103) with $x[n] = \delta[n]$, we get

$$\boxed{y[n] + y[n-1] - 6y[n-2] = x[n],}$$

$$h[0] = 1, \qquad h[1] + h[0] = 0.$$

Solution of the above two sets of equations yields $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$.

Thus, the impulse response is given by

$$\boxed{h[n] = 0.6(-3)^n + 0.4(2)^n, \qquad n \geq 0.}$$

EXAMPLE 2.41 Impulse Response Computation from Total Solution

A causal LTI discrete-time system with an impulse response $h[n]$ satisfies the following difference equation:

$$h[n] - ah[n-1] = \delta[n]. \tag{2.108}$$

We determine a closed-form expression for $h[n]$, and the input–output relation of the above system.

The total solution of the difference equation of Eq. (2.108) is given by

$$h[n] = h_c[n] + h_p[n], \tag{2.109}$$

where $h_c[n]$ and $h_p[n]$ are, respectively, the complementary and the particular solutions. To determine the complementary solution, we set the right-hand side of Eq. (2.108) equal to zero and set $h[n] = \lambda^n$, resulting in

$$\lambda^n - a\lambda^{n-1} = 0.$$

The nontrivial solution of the above equation is $\lambda = a$, and hence, $h_c[n] = a^n$. For the particular solution, we assume $h_p[n] = \beta$. Substituting the expressions for $h_c[n]$ and $h_p[n]$ in Eq. (2.109), we get

$$h[n] = a^n + \beta. \tag{2.110}$$

From Eqs. (2.108) and (2.110), we then have

$$h[0] = 1 = 1 + \beta,$$

implying $\beta = 0$. Hence, the total solution of the difference equation of Eq. (2.108) is given by

$$h[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{2.111}$$

It should be noted that the above result could also have been obtained by induction by evaluating Eq. (2.108) for $n = 0, 1, 2, \ldots$, and then solving for $h[0]$, $h[1]$, $h[2]$, and so on. (Problem 2.44).

To determine the general input–output relation of the above discrete-time system, we convolve both sides of Eq. (2.108) with $x[n]$ and make use of Eq. (2.74) to arrive at

$$y[n] - ay[n-1] = x[n]. \tag{2.112}$$

It follows from the form of the complementary solution given by Eq. (2.102) that the impulse response of a finite-dimensional LTI system characterized by a difference equation of the form of Eq. (2.90) is of infinite length. However, as illustrated in Example 2.42, there exist LTI discrete-time systems with an infinite impulse response that cannot be characterized by the difference equation form of Eq. (2.90).

EXAMPLE 2.42 A Causal Stable LTI Discrete-Time System with No Difference Equation Representation

The system defined by the impulse response

$$h[n] = \frac{1}{n^2}\mu[n-1]$$

does not have a representation in the form of a linear constant coefficient difference equation. It should be noted that the above system is causal and also BIBO stable.

Since the impulse response $h[n]$ of a causal discrete-time system is a causal sequence, Eq. (2.91) can also be used to calculate recursively the impulse response for $n \geq 0$ by setting initial conditions to zero values, that is, by setting $y[-1] = y[-2] = \cdots = y[-N] = 0$, and using a unit sample sequence $\delta[n]$ as

the input $x[n]$. The step response of a causal LTI system can similarly be computed recursively by setting zero initial conditions and applying a unit step sequence as the input. It should be noted that the causal discrete-time system of Eq. (2.91) is linear only for zero initial conditions (Problem 2.67).

2.7.4 Output Computation Using MATLAB

The causal LTI system of the form of Eq. (2.91) can be simulated in MATLAB using the function `filter` already made use of in Program 2_4. The function implements Eq. (2.91) in the form of a set of equations as indicated below:

$$\boxed{y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k],}$$

$$\begin{aligned}
y[n] &= \frac{p_0}{d_0}x[n] + s_1[n-1],\\
s_1[n] &= \frac{p_1}{d_0}x[n] - \frac{d_1}{d_0}y[n] + s_2[n-1],\\
&\vdots\\
s_{N-1}[n] &= \frac{p_{N-1}}{d_0}x[n] - \frac{d_{N-1}}{d_0}y[n] + s_{N-2}[n-1],\\
s_N[n] &= \frac{p_N}{d_0}x[n] - \frac{d_N}{d_0}y[n],
\end{aligned} \tag{2.113}$$

where $s_i[n]$, $1 \le i \le N$, are N internal variables. By back substitution, it can be shown that the above set of equations indeed reduces to Eq. (2.91). The values of the internal variables $s_i[n]$ at the starting instant are called the *initial conditions.*

The basic forms of the function `filter` are as follows:

```
y = filter(p,d,x)
[y, sf] = filter(p,d,x, si]
```

In the first form, the input data vector `x` is processed by the system characterized by the coefficient vectors `p` and `d` to generate the output vector `y`, assuming zero initial conditions. The length of `y` is the same as the length of `x`. The second form permits the inclusion of nonzero initial conditions of the internal variables $s_i[n]$ in the vector `si` and provides an output that includes the vector `sf` as the final values of $s_i[n]$. Since the function implements Eq. (2.91), the coefficient d_0 must be nonzero.

Example 2.43 illustrates the use of the function `filter` in the computation of the total solution.

2.7.5 Impulse and Step Response Computation Using MATLAB

The impulse and step responses of a causal LTI discrete-time system can be computed using the MATLAB M-files `impz` and `stepz`, respectively. Each function is available with several options. We illustrate the use of these two functions in Example 2.44.

EXAMPLE 2.44 Impulse and Step Response Computations Using MATLAB

Determine the first 41 samples of the impulse and response samples of the causal LTI system defined by

$$\begin{aligned} y[n] + 0.7y[n-1] - 0.45y[n-2] - 0.6y[n-3] \\ = 0.8x[n] - 0.44x[n-1] + 0.36x[n-2] + 0.02x[n-3]. \end{aligned} \tag{2.114}$$

The code fragments that can be used to compute the impulse and step response samples are as follows:

```
p = [0.8 -0.44 0.36 0.02];
d = [1 0.7 -0.45 -0.6];
[h,m] = impz(p,d,41);
[s,m] = stepz(p,d,41);
```

The computed first 41 samples of the impulse and step response samples are indicated in Figures 2.36(a) and (b), respectively.

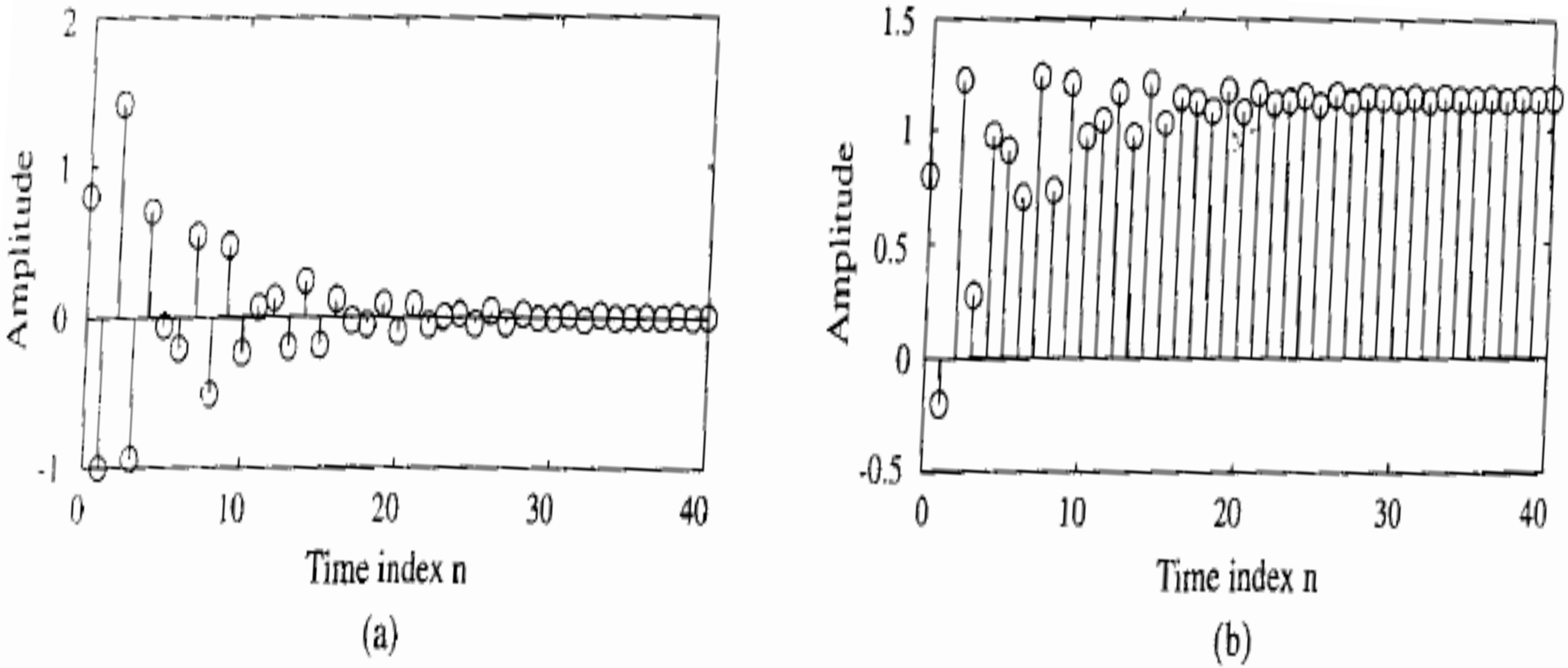

Figure 2.36: (a) Impulse response and (b) step response of the system of Eq. (2.114).

HW#2

due date next lecture, submitted as a hard copy

Q1. Determine the total solution for a discrete time system characterized by the following LCCDE:

$$y[n]=5/6\ y[n-1]-1/6\ y[n-2]+x[n]+\tfrac{1}{2}\ x[n-1]$$

For a step input $x[n]=2^n$, $n\geq 0$

initial conditions $x[-1]=1$, $y[-1]=6$ and $y[-2]=6$

Q2. verify and plot the results of the code of examples 2.43 and 2.44

EXAMPLE 2.37 Total Solution Computation of an LTI System for a Constant Input

Let us determine the total solution for $n \geq 0$ of a discrete-time system characterized by the following difference equation:

$$y[n] + y[n-1] - 6y[n-2] = x[n], \tag{2.103}$$

for a step input $x[n] = 8\mu[n]$ and with initial conditions $y[-1] = 1$ and $y[-2] = -1$.

We first determine the form of the complementary solution. Setting $x[n] = 0$ and $y[n] = \lambda^n$ in Eq. (2.103), we arrive at

$$\begin{aligned}\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} &= \lambda^{n-2}(\lambda^2 + \lambda - 6)\\ &= \lambda^{n-2}(\lambda + 3)(\lambda - 2) = 0,\end{aligned}$$

and hence the roots of the characteristic polynomial $\lambda^2 + \lambda - 6$ are $\lambda_1 = -3$, $\lambda_2 = 2$. Therefore, the complementary solution is of the form

$$y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n. \tag{2.104}$$

For the particular solution, we assume

$$y_p[n] = \beta.$$

Substituting the above in Eq. (2.103), we get

$$\beta + \beta - 6\beta = 8\mu[n],$$

which for $n \geq 0$ yields $\beta = -2$.

The total solution is therefore of the form

$$y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0. \tag{2.105}$$

The constants α_1 and α_2 are chosen to satisfy the specified initial conditions. From Eqs. (2.103) and (2.105), we get

$$\begin{aligned}y[-2] &= \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} - 2 = -1,\\ y[-1] &= \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} - 2 = 1.\end{aligned}$$

Solving these two equations, we arrive at

$$\alpha_1 = -1.8, \quad \alpha_2 = 4.8.$$

Thus, the total solution is given by

$$y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0. \tag{2.106}$$

EXAMPLE 2.38 **Total Solution Computation of an LTI System for an Exponential Input**

We determine the total solution for $n \geq 0$ of the difference equation of Eq. (2.103) for an input $\boxed{x[n] = 2^n \mu[n]}$ with the same initial conditions as in Example 2.37.

As indicated in Example 2.37, the complementary solution contains a term $\alpha_2(2)^n$, which is of the same form as the specified input. Hence, we need to select a form for the particular solution that is distinct and does not contain any terms similar to those contained in the complementary solution. We assume

$$\boxed{y_p[n] = \beta n(2)^n.}$$

Substituting the above in Eq. (2.103), we get

$$\beta n(2)^n + \beta(n-1)(2)^{n-1} - 6\beta(n-2)(2)^{n-2} = (2)^n \mu[n].$$

For $n \geq 0$, we obtain from the above equation $\beta = 0.4$. The total solution is now of the form

$$\boxed{y[n] = \alpha_1(-3)^n + \alpha_2(2)^n + 0.4n(2)^n, \qquad n \geq 0.} \tag{2.107}$$

To determine the values of α_1 and α_2, we make use of the specified initial conditions. From Eqs. (2.103) and (2.107), we arrive at

$$y[-2] = \alpha_1(-3)^{-2} + \alpha_2(2)^{-2} + 0.4(-2)(2)^{-2} = -1,$$
$$y[-1] = \alpha_1(-3)^{-1} + \alpha_2(2)^{-1} + 0.4(-1)(2)^{-1} = 1,$$

which when solved yields $\alpha_1 = -5.04$, $\alpha_2 = -0.96$. Therefore, the total solution is given by

$$\boxed{y[n] = -5.04(-3)^n - 0.96(2)^n + 0.4n(2)^n, \qquad n \geq 0.}$$

Q1. Determine the total solution for a discrete time system characterized by the following LCCDE: $y[n]=5/6y[n-1]-1/6[n-2]+x[n]+1/2x[n-1]$

For a step input $x[n]=2^n, n\geq 0$ and initial conditions $x[-1]=1, y[-1]=6 \; and \; y[-2]=6$

First, determine the form of the complementary solution. Setting x[n]=x[n-1]=0 and y[n]=λ^n. Then we arrive at

$$6\lambda^n-5\lambda^{n-1}+\lambda^{n-2}=\lambda^{n-2}(6\lambda^2-05\lambda+1)=\lambda^{n-2}(3\lambda-1)(2\lambda-1)=0$$

Hence the roots of the characteristic polynomial $6\lambda^2-05\lambda+1$ are $\lambda_1=1/3,\ \lambda_2=1/2$.

Therefore, the complementary solution is of the form

$$\boxed{y_c[n]=\alpha_1(1/3)^n+\alpha_2(1/2)^n}$$

For the particular solution, we assume

$$y_p[n]=\beta 2^n$$

Substituting the equation in the problem, we get

$$6\beta 2^n-5\beta 2^{n-1}+\beta 2^{n-2}=6(2^n+\frac{1}{2}2^{n-1})$$

Which for $n\geq 1$ yields $\beta=2$.

The total solution is therefore of the form

$$\boxed{y[n]=\alpha_1(1/3)^n+\alpha_2(1/2)^n+2\cdot 2^n}$$

The constant α_1, α_2 are chosen to satisfy the modified specified initial conditions.

$$y[0]=\frac{5}{6}y[-1]-\frac{1}{6}y[-2]+x[0]+\frac{1}{2}x[-1] \qquad , \qquad y[1]=\frac{5}{6}y[0]-\frac{1}{6}y[-1]+x[1]+\frac{1}{2}x[0]$$

$$=\frac{5}{6}*6-\frac{1}{6}*6+1+\frac{1}{2}*1=\frac{11}{2} \qquad\qquad =\frac{5}{6}*\frac{11}{2}-\frac{1}{6}*6+2+\frac{1}{2}*1=\frac{73}{12}$$

Solving these two equations, we arrive at $\alpha_1=-2,\ \alpha_2=\frac{11}{2}$

Thus, the total solution is given by

$$y[n]=-2(\frac{1}{3})^n+\frac{11}{2}(\frac{1}{2})^n+2^{n+1},\ n\geq 0$$

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solution pdf file

2.7.2 Zero-Input Response and Zero-State Response

An alternate approach to determining the total solution $y[n]$ of the difference equation of Eq. (2.90) is by computing its *zero-input response*, $y_{zi}[n]$, and *zero-state response*, $y_{zs}[n]$. The component $y_{zi}[n]$ is obtained by solving Eq. (2.90) by setting the input $x[n] = 0$, and the component $y_{zs}[n]$ is obtained by solving Eq. (2.90) by applying the specified input with all initial conditions set to zero. The total solution is then given by $y_{zi}[n] + y_{zs}[n]$.

This approach is illustrated in Example 2.39.

EXAMPLE 2.39 Total Solution Computation from Zero-Input and Zero-State Responses

We determine the total solution of the discrete-time system of Example 2.37 by computing the zero-input response and the zero-state response.

$$\boxed{y[n] + y[n-1] - 6y[n-2] = x[n],}$$

The zero-input response, $y_{zi}[n]$, of Eq. (2.103) is given by the complementary solution of Eq. (2.104), where the constants α_1 and α_2 are chosen to satisfy the specified initial conditions. Now, from Eq. (2.103), we get

$$y[0] = -y[-1] + 6y[-2] = -1 - 6 = -7, \qquad y[1] = -y[0] + 6y[-1] = 7 + 6 = 13.$$

Next, from Eq. (2.104), we get $\boxed{y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n.}$

$$y[0] = \alpha_1 + \alpha_2, \qquad y[1] = -3\alpha_1 + 2\alpha_2.$$

Solving these two sets of equations, we arrive at $\alpha_1 = -5.4$, $\alpha_2 = -1.6$. Therefore,

$$\boxed{y_{zi}[n] = -5.4(-3)^n - 1.6(2)^n, \qquad n \geq 0.}$$

The zero-state response is determined from Eq. (2.105) by evaluating the constants α_1 and α_2 to satisfy the zero initial conditions. From Eq. (2.103), we get $\boxed{y[n] = \alpha_1(-3)^n + \alpha_2(2)^n - 2, \qquad n \geq 0.}$

$$y[0] = x[0] = 8, \qquad y[1] = x[1] - y[0] = 0.$$

Next, from Eq. (2.105) and the above set of equations, we arrive at $\alpha_1 = 3.6$, $\alpha_2 = 6.4$. Thus, the zero-state response for $n \geq 0$ with initial conditions $y_{zs}[-2] = y_{zs}[-1] = 0$ is given by

$$y_{zs}[n] = 3.6(-3)^n + 6.4(2)^n - 2.$$

Hence, the total solution $y[n]$ is given by the sum $y_{zi}[n] + y_{zs}[n]$, resulting in

$$\boxed{y[n] = -1.8(-3)^n + 4.8(2)^n - 2, \qquad n \geq 0,}$$

which is identical to that derived in Example 2.37, as expected.

Total solution computation from zero input and zero state response

Example

$$\boxed{y_n = \frac{5}{6} y_{n-1} - \frac{1}{6} y_{n-2} + x_n + \frac{1}{2} x_{n-1}}$$

$$\left.\begin{array}{l} y_{-1} = 6 \\ y_{-2} = 6 \\ x_{-1} = 1 \end{array}\right\}$$ actually in SFG form!

$x_n = 2^n, \; n \geq 0$

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{3}\right)$$

$$\Downarrow$$

$$\lambda_1 = \frac{1}{2}$$

$$\lambda_2 = \frac{1}{3}$$

$$\Downarrow$$

$$\boxed{y_{h,n} = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n}$$ FORM

Home work example (from A.A.Beex lecture notes)

zero input response

$y_{zi,n}$? $\rightarrow$ assume zero input $\Rightarrow$ $y_{p,n} = c$

$\Downarrow$

$c - \frac{5}{6}c + \frac{1}{6}c = 0 + 0$

$\Downarrow$

$c = 0$

$\Downarrow$

$y_{p,n} = 0$

$$\boxed{y_{zi,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n \qquad \text{form of } y_{h,n}}$$

$n=0: \; y_0 = \frac{5}{6}\cancel{y_{-1}}^{\,6} - \frac{1}{6}\cancel{y_{-2}}^{\,6} + \cancel{x_0}^{\,0} + \frac{1}{2}\cancel{x_{-1}}^{\,1} = 5 - 1 + \frac{1}{2} = 4\frac{1}{2} = \left(\frac{1}{2}\right)^0 c_1 + \left(\frac{1}{3}\right)^0 c_2$

$n=1: \; y_1 = \frac{5}{6}\cancel{y_0}^{\,4\frac{1}{2}} - \frac{1}{6}\cancel{y_{-1}}^{\,6} + \cancel{x_1}^{\,0} + \frac{1}{2}\cancel{x_0}^{\,0} = \frac{22.5}{6} - 1 = \left(\frac{1}{2}\right)^1 c_1 + \left(\frac{1}{3}\right) c_2$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \boxed{y_{zi,n} = 7.5\left(\tfrac{1}{2}\right)^n - 3\left(\tfrac{1}{3}\right)^n ; \quad n \geq 0}$$

Home work example (from A.A.Beex lecture notes)

zero state response

$y_{zs,n}$?

from before : $y_{p,n} = 2 \cdot 2^n$; $n \geq 0$

like a $y_{TOT,n}$ but for i.c.'s $\equiv 0$

$$\boxed{y_{zs,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n + 2 \cdot 2^n}$$

$n=0$: $\frac{5}{6}\cancel{y_{-1}}^{0} - \frac{1}{6}\cancel{y_{-2}}^{0} + \cancel{x_0}^{1} + \frac{1}{2}\cancel{x_{-1}}^{0} = 1 \quad = c_1 + c_2 + 2$

$n=1$: $\frac{5}{6}\cancel{y_0}^{1} - \frac{1}{6}\cancel{y_{-1}}^{0} + \cancel{x_1}^{2} + \frac{1}{2}\cancel{x_0}^{1} = \frac{5}{6} + 2.5 = c_1 \cdot \frac{1}{2} + c_2 \cdot \frac{1}{3} + 4$

$y_2 = \frac{5}{6} y_1 - \frac{1}{6}\cancel{y_0}^{1} + \cancel{x_2}^{4} + \frac{1}{2}\cancel{x_1}^{2} = \cdots = c_1\left(\frac{1}{2}\right)^2 + c_2\left(\frac{1}{3}\right)^2 + 8$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\Downarrow$

$$\underline{c} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{y_{zs,n} = (-2)\left(\tfrac{1}{2}\right)^n + 1 \cdot \left(\tfrac{1}{3}\right)^n + 2 \cdot 2^n \quad ; \quad n \geq 0}$$

$y_{ss,n}$

steady state

Note : $y_{zs,n} + y_{zi,n} = \underbrace{(-2+7.5)}_{5.5}\left(\frac{1}{2}\right)^n + \underbrace{(1-3)}_{-2}\left(\frac{1}{3}\right)^n + 2 \cdot 2^n$; $n \geq 0$

$= y_n = y_{TOT,n}$!

the rest is $y_{transient,n}$

$\left(= 5.5\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n ; n \geq 0\right)$

Home work example (from A.A.Beex lecture notes)

2.7.3 Impulse Response Calculation

The impulse response $h[n]$ of a causal LTI discrete-time system is the output observed with input $x[n] = \delta[n]$. Thus, it is simply the zero-state response with $x[n] = \delta[n]$. Now for such an input, $x[n] = 0$ for $n > 0$, and thus, the particular solution is zero, that is, $y_p[n] = 0$. Hence, the impulse response can be computed from the complementary solution of Eq. (2.101) in the case of simple roots of the characteristic equation by determining the constants α_i to satisfy the zero initial conditions. A similar procedure can be followed in the case of multiple roots of the characteristic equation. A system with all zero initial conditions is often called a *relaxed* system.

We illustrate the impulse response computation in Examples 2.40 and 2.41.

$$\boxed{h_n\,? = y_{TOT,n} \text{ for i.c.'s} \equiv 0 \text{ and } x_n = \delta_n}$$

$$y_{h,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n \quad \underline{\text{as before}} \quad \text{(check it out)}$$

$$\Rightarrow \quad h_n = y_{TOT,n} = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(\tfrac{1}{3}\right)^n + K\delta_n \;;\; n \geq 0$$

$$n=0: \quad h_0 = \tfrac{5}{6}(0) - \tfrac{1}{6}(0) + \cancel{x_0}^{\,1} + \tfrac{1}{2}\cancel{x_{-1}}^{\,0} = 1 = c_1 + c_2 + K$$

$$n=1: \quad h_1 = \tfrac{5}{6}\cancel{h_0}^{\,1} - \tfrac{1}{6}(0) + \cancel{x_1}^{\,0} + \tfrac{1}{2}\cancel{x_0}^{\,1} = \tfrac{5}{6} + \tfrac{1}{2} = c_1\cdot\tfrac{1}{2} + c_2\cdot\tfrac{1}{3} + 0$$

$$n=2: \quad h_2 = \tfrac{5}{6}\underbrace{\left(\tfrac{5}{6}+\tfrac{1}{2}\right)}_{h_1} - \tfrac{1}{6}\underbrace{(1)}_{h_0} + \cancel{x_2}^{\,0} + \tfrac{1}{2}\cancel{x_1}^{\,0} = \quad = c_1\left(\tfrac{1}{2}\right)^2 + c_2\left(\tfrac{1}{3}\right)^2 + 0$$

$$\boxed{y_n = \tfrac{5}{6}y_{n-1} - \tfrac{1}{6}y_{n-2} + x_n + \tfrac{1}{2}x_{n-1}}$$

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ K \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix}$$

$$\boxed{h_n = 6\left(\tfrac{1}{2}\right)^n - 5\left(\tfrac{1}{3}\right)^n \;;\; n \geq 0}$$

it's causal

$$h_n = \left[6\left(\tfrac{1}{2}\right)^n - 5\left(\tfrac{1}{3}\right)^n\right]u_n$$

lasts forever $\Rightarrow$ IIR

$\sum_n |h_n| < \infty \Leftrightarrow$ BIBO

Home work example (from A.A.Beex lecture notes)

EXAMPLE 2.40 Impulse Response Computation from Zero-State Response

In this example, we determine the impulse response $h[n]$ of the causal discrete-time system of Example 2.37. From Eq. (2.104), we get

$$\boxed{h[n] = \alpha_1(-3)^n + \alpha_2(2)^n, \qquad n \geq 0.}$$

From the above, we arrive at

$$h[0] = \alpha_1 + \alpha_2, \qquad h[1] = -3\alpha_1 + 2\alpha_2.$$

Next, from Eq. (2.103) with $x[n] = \delta[n]$, we get $\boxed{y[n] + y[n-1] - 6y[n-2] = x[n],}$

$$h[0] = 1, \qquad h[1] + h[0] = 0.$$

Solution of the above two sets of equations yields $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$.

Thus, the impulse response is given by

$$\boxed{h[n] = 0.6(-3)^n + 0.4(2)^n, \qquad n \geq 0.}$$

2.7.4 Output Computation Using MATLAB

The causal LTI system of the form of Eq. (2.91) can be simulated in MATLAB using the function `filter` already made use of in Program 2_4. The function implements Eq. (2.91) in the form of a set of equations as indicated below:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k],$$

$$\begin{aligned}
y[n] &= \frac{p_0}{d_0}x[n] + s_1[n-1],\\
s_1[n] &= \frac{p_1}{d_0}x[n] - \frac{d_1}{d_0}y[n] + s_2[n-1],\\
&\vdots\\
s_{N-1}[n] &= \frac{p_{N-1}}{d_0}x[n] - \frac{d_{N-1}}{d_0}y[n] + s_{N-2}[n-1],\\
s_N[n] &= \frac{p_N}{d_0}x[n] - \frac{d_N}{d_0}y[n],
\end{aligned} \tag{2.113}$$

where $s_i[n]$, $1 \le i \le N$, are N internal variables. By back substitution, it can be shown that the above set of equations indeed reduces to Eq. (2.91). The values of the internal variables $s_i[n]$ at the starting instant are called the *initial conditions.*

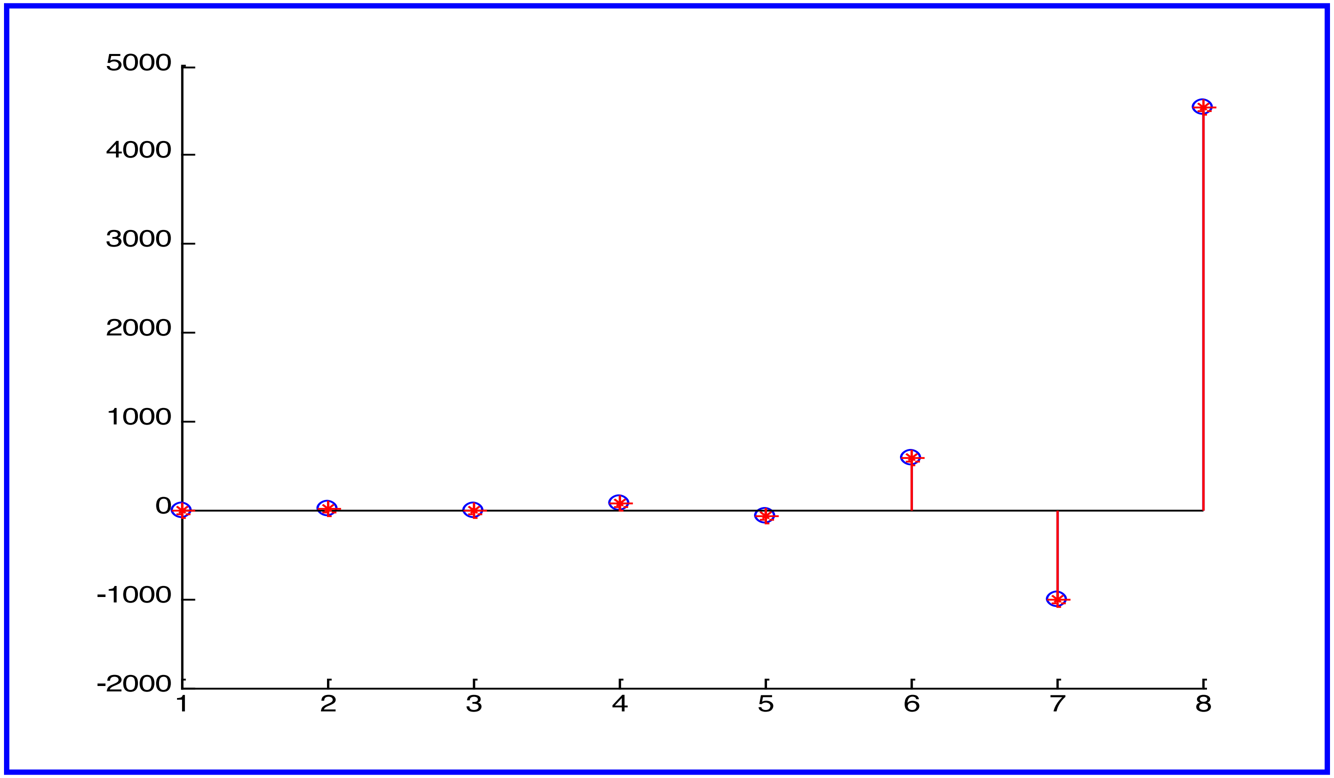
The basic forms of the function `filter` are as follows:

```
y = filter(p,d,x)
[y, sf] = filter(p,d,x, si]
```

In the first form, the input data vector `x` is processed by the system characterized by the coefficient vectors `p` and `d` to generate the output vector `y`, assuming zero initial conditions. The length of `y` is the same as the length of `x`. The second form permits the inclusion of nonzero initial conditions of the internal variables $s_i[n]$ in the vector `si` and provides an output that includes the vector `sf` as the final values of $s_i[n]$. Since the function implements Eq. (2.91), the coefficient d_0 must be nonzero.

Example 2.43 illustrates the use of the function `filter` in the computation of the total solution.

```
[y1,sf]=filter(1,[1,1,-6],8*ones(1,8),[-7,6]);
stem(y1)
hold on;
n=0:7;
y2=-1.8*(-3).^n+4.8*(2).^n-2;
stem(y2,'r*')
```



2.7.5 Impulse and Step Response Computation Using MATLAB

The impulse and step responses of a causal LTI discrete-time system can be computed using the MATLAB M-files `impz` and `stepz`, respectively. Each function is available with several options. We illustrate the use of these two functions in Example 2.44.

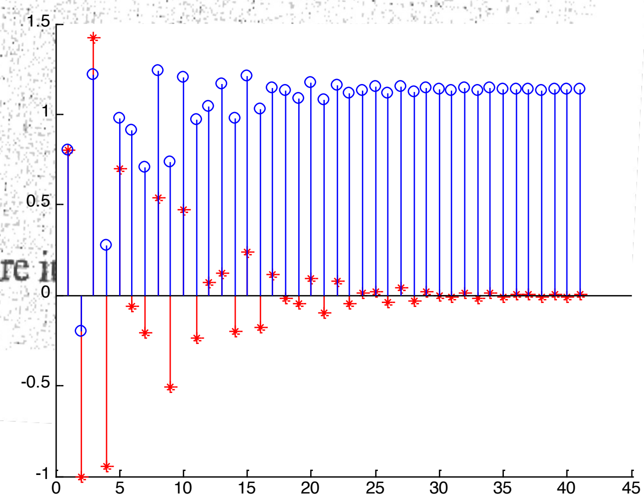
EXAMPLE 2.44 Impulse and Step Response Computations Using MATLAB

Determine the first 41 samples of the impulse and response samples of the causal LTI system defined by

$$\boxed{\begin{aligned} y[n] + 0.7y[n-1] - 0.45y[n-2] - 0.6y[n-3] \\ = 0.8x[n] - 0.44x[n-1] + 0.36x[n-2] + 0.02x[n-3]. \end{aligned}} \tag{2.114}$$

The code fragments that can be used to compute the impulse and step response samples are as follows:

```
p = [0.8 -0.44 0.36 0.02];
d = [1 0.7 -0.45 -0.6];
[h,m] = impz(p,d,41);
[s,m] = stepz(p,d,41);
```



The computed first 41 samples of the impulse and step response samples are i
respectively.

2.7.6 Location of Roots of Characteristic Equation for BIBO Stability

It should be noted that the impulse response samples of a stable LTI system decay to zero values as the time index n becomes very large. Likewise, the step response samples of a stable LTI system approach a constant value as n becomes very large. From the plots of Figure 2.36(a) and (b), we can conclude that most likely the LTI system of Eq. (2.114) is BIBO stable. However, it is impossible to check the stability of a system just by examining only a finite segment of its impulse or step response as in these figures.

The BIBO stability of a causal LTI system characterized by a constant coefficient difference equation of the form of Eq. (2.90) can be inferred from the values of the roots λ_i of its characteristic polynomial. To establish the stability conditions, recall that the form of the impulse response is the same as that of the complementary solution. From Eq. (2.101), assuming all the roots to be distinct, we have

$$h[n] = \sum_{i=1}^{N} \alpha_i \lambda_i^n \mu[n]. \tag{2.115}$$

The constants α_i in the above expression are determined to satisfy zero initial conditions. From Eq. (2.115) we get

$$\boxed{\sum_{n=0}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left| \sum_{i=1}^{N} \alpha_i (\lambda_i)^n \right| \leq \sum_{i=1}^{N} |\alpha_i| \sum_{n=0}^{\infty} |\lambda_i|^n.} \tag{2.116}$$

It follows from the above equation that if $|\lambda_i| < 1$ for all values of i, then $\sum_{n=0}^{\infty} |\lambda_i|^n < \infty$, and as a result, $\sum_{n=0}^{\infty} |h[n]| < \infty$; that is, the impulse response is absolutely summable, implying BIBO stability of the causal LTI discrete-time system. However, the impulse response sequence is not absolutely summable if one or more of the roots λ_i has a magnitude greater than or equal to one. It should be noted that the discrete-time system of Example 2.37 described in Eq. (2.103) is clearly an unstable system as both roots of the characteristic equation have magnitudes greater than one.

In the case of multiple roots of the characteristic equation, the impulse response will contain terms of the form $n^K \lambda_i^n$. As a result, the expression for $\sum_{n=0}^{\infty} |h[n]|$ will contain the term

$$\sum_{n=0}^{\infty} |n^K (\lambda_i)^n|,$$

which converges if $|\lambda_i| < 1$ (Problem 2.89), and as a result, here also the impulse response is absolutely summable.

Summarizing, a causal LTI system characterized by a linear constant coefficient difference equation of the form of Eq. (2.90) is BIBO stable if the magnitude of each of the roots of its characteristic equation is less than one. This condition is both necessary and sufficient.

2.8 Classification of LTI Discrete-Time Systems

Linear time-invariant (LTI) discrete-time systems are usually classified either according to the length of their impulse response sequences or according to the method of calculation employed to determine the output samples.

2.8.1 Classification Based on Impulse Response Length

If $h[n]$ is of finite length, that is,

$$h[n] = 0 \quad \text{for} \quad n < N_1 \quad \text{and} \quad n > N_2 \quad \text{with} \quad N_1 < N_2, \tag{2.117}$$

then it is known as a *finite impulse response* (FIR) discrete-time system, for which the convolution sum reduces to

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]. \tag{2.118}$$

Note that the above convolution sum, being a finite sum, can be used to calculate $y[n]$ directly. The basic operations involved are simply multiplication and addition. Note that the calculation of the present value of the output sequence involves the value of the input sample at $n = N_1$ and $N_2 - N_1$ previous values of the input sequence along with the $N_2 - N_1 + 1$ impulse response samples describing the FIR discrete-time system.

Examples of FIR discrete-time systems are the moving-average system of Eq. (2.61) and the linear interpolators of Eqs. (2.65) and (2.66).

If $h[n]$ is of infinite length, then it is known as an *infinite impulse response* (IIR) discrete-time system. For a causal IIR discrete-time system with a causal input $x[n]$, the convolution sum can be expressed in the form

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k],$$

which can be used to compute the output samples. However, for increasing n, the computational complexity to compute the output sample increases as the number of products to be summed also increases.

Chapter (2) - Checklist

- 2.1 Discrete time signal √
- 2.2 Typical sequences and sequence representation√
- 2.3 The sampling Process√
- 2.4 Discrete Time systems√
- 2.5 Time Domain characterization of LTI Discrete-Time systems √
- 2.6 Simple interconnection schemes√
- 2.7 Finite-Dimensional LTI Discrete time systems√
- 2.8 classification of LTI Discrete time systems √

Home work # (3)

- Reading section correlation of signals:
  - Only 2.9.1 and 3 , examples 2.46 and 2.47
- Solve proplems of chapter 2 page 107-115:
  - # 1, 5, 6, 7(a and c), 8, 17, 25, 38, 50, 64,83,90
- Implement matlab exercise page 115:
  - # 9

Notes:

- Some final answers will be posted on the course web page
- Submit it as a hardcopy.
- Due date is 07 Feb ,