EENG 479 : Digital Signal Processing (DSP)

Lecture #7:

Chapter 6: Z Transform

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z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \left|e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  leads to the z-transform

- z-transform may exist for many sequences for which the DTFT does not exist

- Moreover, use of z-transform techniques permits simple algebraic manipulations

z-Transform

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where $z = \mathcal{R}e(z) + j\,\mathcal{I}m(z)$ is a complex variable

z-Transform

- If we let $z = r\,e^{j\omega}$, then the z-transform reduces to

$$G(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

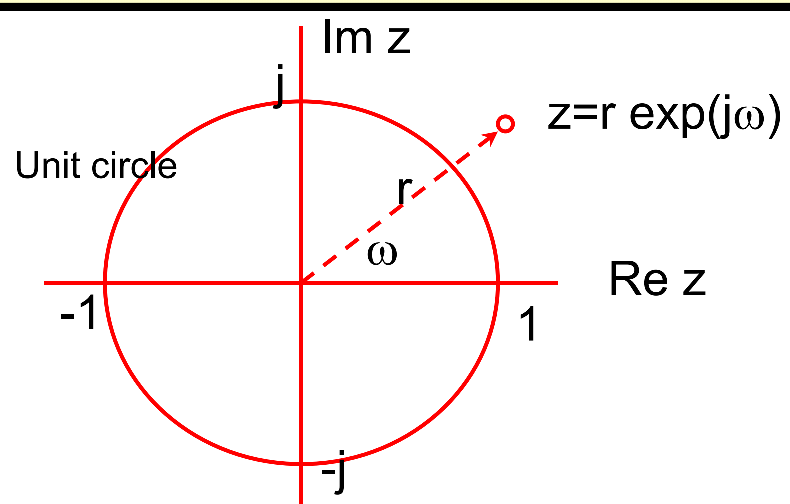
- The above can be interpreted as the DTFT of the modified sequence $\{g[n] r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z-transform reduces to its DTFT, provided the latter exists

z-Transform

- The contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the **unit circle**
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set $\mathcal{R}$ of values of z for which its z-transform converges is called the **region of convergence** (ROC)



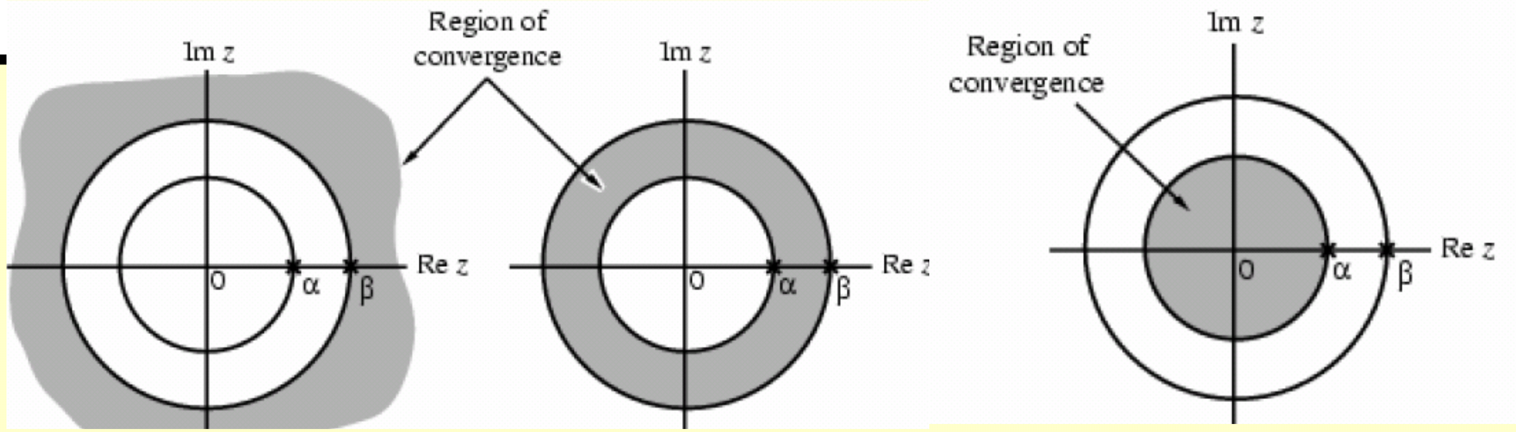
z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left|g[n]r^{-n}\right| < \infty$$



- In general, the ROC $\mathcal{R}$ of a z-transform of a sequence $g[n]$ is an annular region of the z-plane:

$$R_{g^-} < |z| < R_{g^+}$$

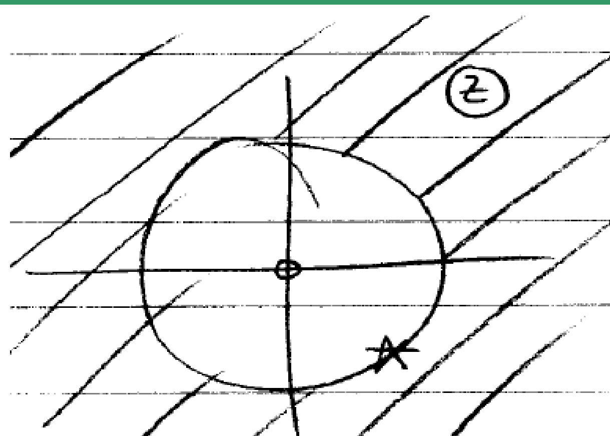
where $0 \le R_{g^-} < R_{g^+} \le \infty$

- Note: The z-transform is a form of a Laurent series and is an analytic function at every point in the ROC

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1 \rightarrow \frac{z}{z-\alpha}$$

- ROC is the annular region $|z| > |\alpha|$



$= \frac{N(z)}{D(z)}$ $\rightarrow N(z)=0 \rightarrow$ zero; $\rightarrow D(z)=0 \rightarrow$ pole

- Example - The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z^{-1}| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

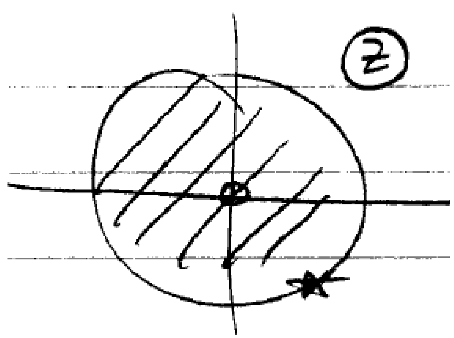
$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^{m}$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^{m} = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1 \;\rightarrow\; \frac{z}{z-\alpha}$$

- ROC is the annular region $|z| < |\alpha|$

Note ∃ pole on ROC boundary



z-Transform

- <u>Note</u>: The z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

Table 6.1: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent *z*-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational z-Transforms

- The **degree** of the numerator polynomial $P(z)$ is M and the **degree** of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational z-Transforms

- A rational z-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

Gain constant

$H(z) = \sum_{m=1}^{M} b_m z^{-m}$ "all-zero filter"

$H(z) = \frac{C}{\sum_{k=1}^{N} a_k z^{-k}}$ "all-pole filter"

mathematically: # poles = # zeros

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the **zeros** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the **poles** of $G(z)$

Rational z-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

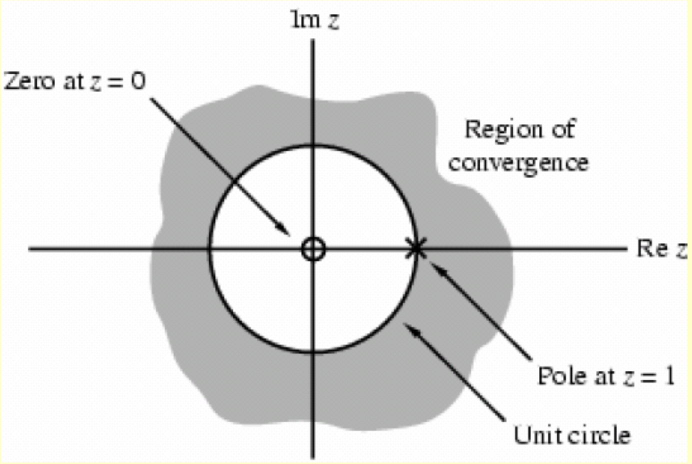
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational *z*-Transforms

- Example - The *z*-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

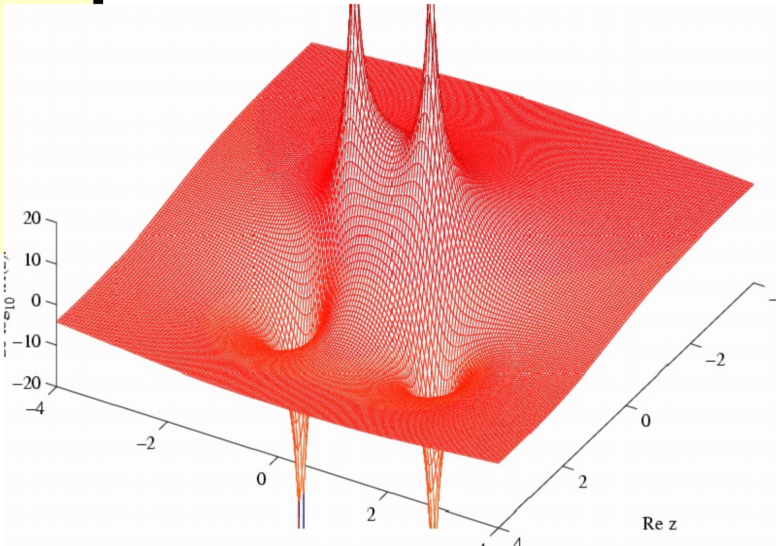
has a zero at $z = 0$ and a pole at $z = 1$



- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$



ROC of a Rational z-Transform

- ROC of a z-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform
- Hence, the z-transform must always be specified with its ROC
- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle

LTI, BIBO $\Longleftrightarrow$ $\sum_n |h_n| < \infty$ $\Longrightarrow$ $\sum_n h_n z^{-n} < \infty$ for $|z| = 1$

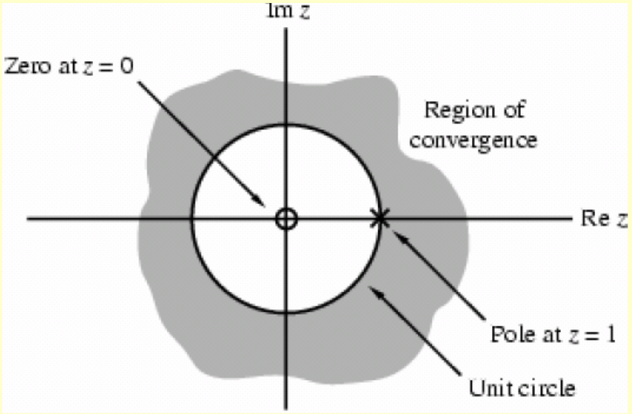
i.e. $|z|=1 \in ROC_H$
u.c.

BIBO relates to whether $|z|=1 \in ROC$ $\rightarrow$ $h_n = a^n u_n$ is BIBO if $|a|<1$

$\downarrow$

$h_n = -a^n u_{-n-1}$ is BIBO if $|a|>1$

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$



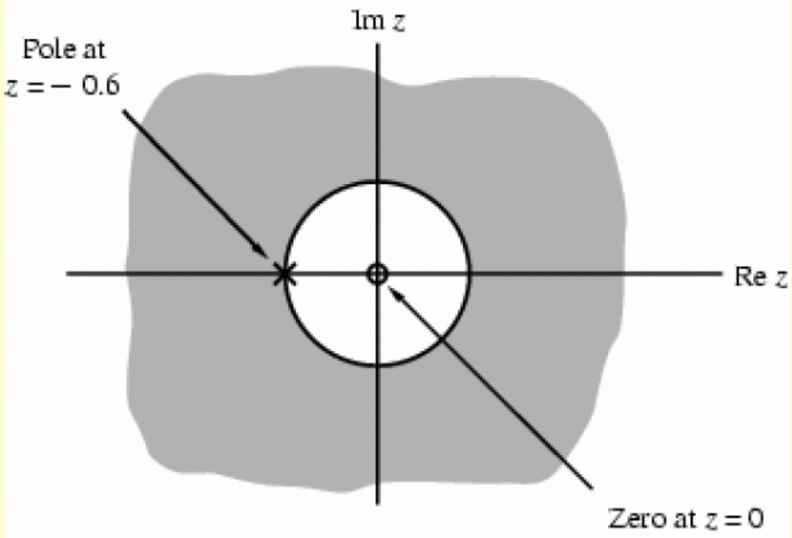
- In this plot, the ROC, shown as the shaded area, is the region of the z-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1 + 0.6\, z^{-1}},$$

$$|z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational z-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

- Example - A **right-sided sequence** with nonzero sample values for $n \geq 0$ is sometimes called a **causal sequence**
- Consider a causal sequence $u_1[n]$
- Its z-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

- Example - A **left-sided sequence** with nonzero sample values for $n \leq 0$ is sometimes called a **anticausal sequence**
- Consider an anticausal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N nonnegative has a z-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

- Example - The z-transform of a two-sided sequence $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$
- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n] z^{-n}$, can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the z-transform does not exist

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

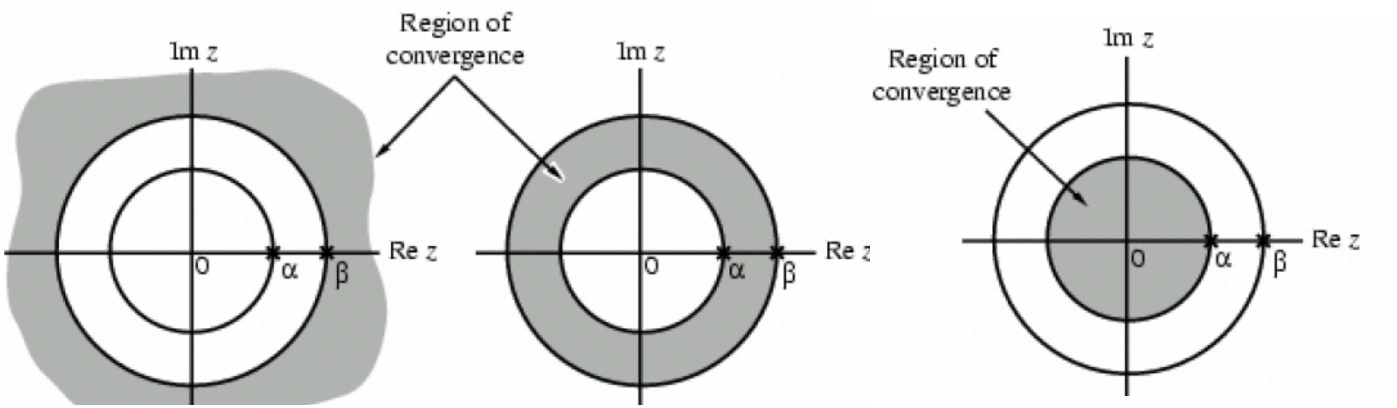
- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$
- There is no overlap between these two regions
- Hence, the z-transform of $u[n] = \alpha^n$ does not exist

There are three possible ROCs of a rational z-transform with poles at $z = \alpha$ and $z = \beta$ ($|\alpha| < |\beta|$)

In general, if the rational z-transform has N poles with R distinct magnitudes, then it has $R+1$ ROCs

Thus, there are $R+1$ distinct sequences with the same z-transform

Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform



- The ROC of a rational *z*-transform can be easily determined using MATLAB

```
[z,p,k] = tf2zp(num,den)
```

determines the zeros, poles, and the gain constant of a rational *z*-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

```
[num,den] = zp2tf(z,p,k)
```

implements the reverse process

The factored form of the *z*-transform can be obtained using `sos = zp2sos(z,p,k)`

The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix `sos`

$$sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

- The pole-zero plot is determined using the function `zplane`
- The z-transform can be either described in terms of its `zeros` and `poles`:

`zplane(zeros,poles)`

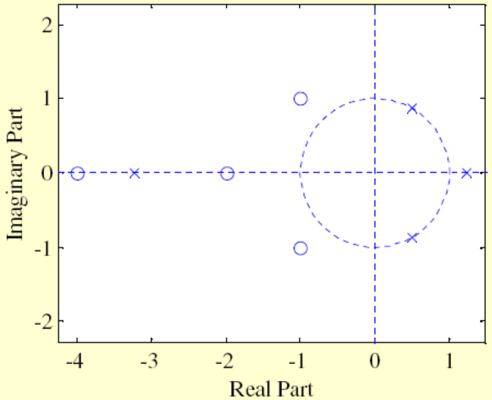
- or, it can be described in terms of its numerator and denominator coefficients:

`zplane(num,den)`

Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below



× – pole
o – zero

assignment

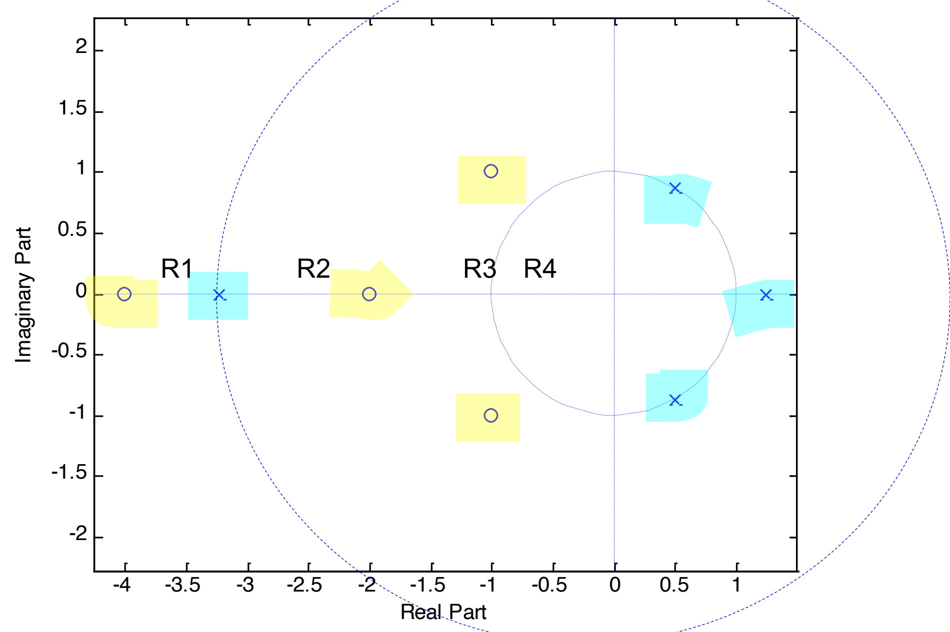
factorize.m
Prog 6_1.m
Prog 6_2.m

```
% Program 6_1
% Determination of the Factored Form
% of a Rational z-Transform
%
num = input('Type in the numerator coefficients = ');
den = input('Type in the denominator coefficients = ');

K = num(1)/den(1);

Numfactors = factorize(num)
Denfactors = factorize(den)

disp('Numerator factors');disp(Numfactors);
disp('Denominator factors');disp(Denfactors);
disp('Gain constant');disp(K);

zplane(num,den)
```

$$\frac{2z^4+16z^3+44z^2+56z+32}{3z^4+3z^3-15z^2+18z-12}$$

```
Type in the numerator coefficients = [2 16 44 56 32]
Type in the denominator coefficients = [3 3 -15 18 -12]
Numfactors =
   1.0000    4.0000         0
   1.0000    2.0000         0
   1.0000    2.0000    2.0000
Denfactords=
   1.0000    3.2361         0
   1.0000   -1.2361         0
   1.0000   -1.0000    1.0000
Gain constant
   0.6667
```

$$\frac{0.6667(1+4z^{-1})(1+2z^{-1})(1+2z^{-1}+2z^{-2})}{(1+3.236z^{-1})(1-1.236z^{-1})(1-z^{-1}+z^{-2})}$$



```
% Program 6_2
% Determination of the Rational z-Transform
% from its Poles and Zeros
%
format long

zr = input('Type in the zeros as a row vector = ');
pr = input('Type in the poles  as a row vector = ');
% Transpose zero and pole row vectors
z = zr';
p = pr';

k = input('Type in the gain constant = ');

[num, den] = zp2tf(z, p, k);

disp('Numerator polynomial coefficients');
disp(num);
disp('Denominator polynomial coefficients');
disp(den);
```

```
Type in the zeros as a row vector =

 [0.21 3.14 -0.3+j*0.5 -0.3-j*0.5]

Type in the poles  as a row vector = >>

 [-0.45 0.67  0.81+j*0.72 0.81+j*0.72]

Type in the gain constant = >> 2.2

Numerator polynomial coefficients
2.2  -6.05 -2.22  -1.635 0.49

Denominator polynomial coefficients
   1.0  -1.84 0.19  0.458 -0.0415
```