EENG 479 : Digital Signal Processing (DSP)

Lecture #8:

Chapter 6 : Inverse Z Transform

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Table 6.1: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Inverse z-Transform

An alternative approach to the implementation of the convolution sum is :

to form the product of the z-transforms of the individual sequences being convolved and then evaluating the inverse z-transform of the product.

In many applications this approach is more convenient as it leads to a closed form answer

Thus how to compute inverse z transform

(1) Cauchy's Residue Theorem
(2) Table Look Up Method
(3) Partial Fraction Method
(4) Long Division

General Expression: Recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g[n]r^{-n}$

Accordingly, the inverse DTFT is thus given by

$$g[n]r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{j\omega}) e^{j\omega n} d\omega$$

By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z) z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

But the integral remains unchanged when is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$

The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum \begin{bmatrix} \text{residues of } G(z)z^{n-1} \\ \text{at the poles inside } C \end{bmatrix}$$

The above equation needs to be evaluated at all values of n and is not pursued here

As it is difficult to arrive to close form for it. Thus other methods are used

A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle

Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

(B) Look up tables and recognition

$$X(z) = e^{a/z}, \quad |z| > 0$$

$$= \sum_{k\geq 0}^{\infty} \left(\frac{a}{z}\right)^k \Big/ k! \quad \Rightarrow \quad x_n = \frac{a^n}{n!} u_n$$

$$X(z) = \log(1 - a z^{-1}) = -\sum_{n\geq 1}^{\infty} \frac{a^n}{n} z^{-n}; \quad |z| > |a|$$

$$\Downarrow$$

$$x_n = \frac{-a^n}{n} u_{n-1}$$

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**

- Example - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Reminder

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**

- **Simple Poles**: In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of $G(z)$ be at $z = \lambda_k, 1 \le k \le N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

- The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) G(z) \big|_{z=\lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_\ell|$ and, thus has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$
- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

- Now

$$\rho_1 = (1-0.2z^{-1})H(z)\big|_{z=0.2} = \left.\frac{1+2z^{-1}}{1+0.6z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1+0.6z^{-1})H(z)\big|_{z=-0.6} = \left.\frac{1+2z^{-1}}{1-0.2z^{-1}}\right|_{z=-0.6} = -1.75$$

- Hence

$$H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- **Multiple Poles:** If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N-L$ poles be simple and at $z = \lambda_\ell$, $1 \le \ell \le N-L$

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z)\right]_{z=\nu},$$

$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division
- Example - Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result

$$\{h[n]\} = \{1 \quad 1.6 \quad -0.52 \quad 0.4 \quad -0.2224 \quad \cdots\}, \quad n \geq 0$$

- `[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors `num` and `den`
- Vector `r` contains the residues
- Vector `p` contains the poles
- Vector `k` contains the constants η_ℓ

- `[num,den]=residuez(r,p,k)` converts a z-transform expressed in a partial-fraction expansion form to its rational form

- The function `impz` can be used to find the inverse of a rational z-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

```
%Program 6_3
% Partial-Fraction Expansion of Rational z-Transform
%
num = input('Type in numerator coefficients = ');
den = input('Type in denominator coefficients = ');
[r,p,k] = residuez(num,den);
disp('Residues');disp(r')
disp('Poles');disp(p')
disp('Constants');disp(k)
```

```
% Program 6_4
% Partial-Fraction Expansion to Rational z-Transform
%
r = input('Type in the residues = ');
p = input('Type in the poles = ');
k = input('Type in the constants = ');
[num, den] = residuez(r,p,k);
disp('Numerator polynomial coefficients'); disp(num)
disp('Denominator polynomial coefficients'); disp(den)
```

```
% Program 6_5
% Power Series Expansion of a Rational z-Transform
%
% Read in the number of inverse z-transform coefficients to be computed
L = input('Type in the length of output vector = ');
% Read in the numerator and denominator coefficients of
% the z-transform
num = input('Type in the numerator coefficients = ');
den = input('Type in the denominator coefficients = ');
% Compute the desired number of inverse transform coefficients
[y,t] = impz(num,den,L);
disp('Coefficients of the power series expansion');
disp(y')
```

Table 6.2: z-Transform Properties

Property	Sequence	z -Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v)H(z/v)v^{-1}\, dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1}\, dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g \mathcal{R}_h$ denotes the region $R_{g-} R_{h-} < |z| < R_{g+} R_{h+}$.

LTI Discrete-Time Systems in the Transform Domain

- An LTI discrete-time system is completely characterized in the time-domain by its impulse response sequence $\{h[n]\}$
- Thus, the transform-domain representation of a discrete-time signal can also be equally applied to the transform-domain representation of an LTI discrete-time system

LTI Discrete-Time Systems in the Transform Domain

- Such transform-domain representations provide additional insight into the behavior of such systems
- It is easier to design and implement these systems in the transform-domain for certain applications
- We consider now the use of the DTFT and the z-transform in developing the transform-domain representations of an LTI system

Finite-Dimensional LTI Discrete-Time Systems

- In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

Finite-Dimensional LTI Discrete-Time Systems

- Applying the z-transform to both sides of the difference equation and making use of the linearity and the time-invariance properties of Table 6.2 we arrive at

$$\sum_{k=0}^{N} d_k z^{-k} Y(z) = \sum_{k=0}^{M} p_k z^{-k} X(z)$$

where $Y(z)$ and $X(z)$ denote the z-transforms of $y[n]$ and $x[n]$ with associated ROCs, respectively

Finite-Dimensional LTI Discrete-Time Systems

- A more convenient form of the z-domain representation of the difference equation is given by

$$\left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)$$

The Transfer Function

- A generalization of the frequency response function
- The convolution sum description of an LTI discrete-time system with an impulse response $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Taking the z-transforms of both sides we get

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{n=-\infty}^{\infty} x[n-k]z^{-n}\right)$$

$$= \sum_{k=-\infty}^{\infty} h[k]\left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+k)}\right)$$

- Or, $Y(z) = \sum_{k=-\infty}^{\infty} h[k]\underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell}\right)}_{X(z)}z^{-k}$

- Therefore, $Y(z) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)}_{H(z)}X(z)$

- Thus, $Y(z) = H(z)X(z)$

- Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Its transfer function is obtained by taking the z-transform of both sides of the above equation
- Thus

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- Or, equivalently as

$$H(z) = z^{(N-M)} \frac{\sum_{k=0}^{M} p_k z^{M-k}}{\sum_{k=0}^{N} d_k z^{N-k}}$$

- An alternate form of the transfer function is given by

$$H(z) = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^{M} (1 - \xi_k z^{-1})}{\prod_{k=1}^{N} (1 - \lambda_k z^{-1})}$$

- Or, equivalently as

$$H(z)=\frac{p_0}{d_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

- $\xi_1,\xi_2,\ldots,\xi_M$ are the finite **zeros**, and $\lambda_1,\lambda_2,\ldots,\lambda_N$ are the finite **poles** of $H(z)$
- If $N>M$, there are additional $(N-M)$ zeros at $z=0$
- If $N<M$, there are additional $(M-N)$ poles at $z=0$
- For a causal IIR digital filter, the impulse response is a causal sequence
- The ROC of the causal transfer function

$$H(z)=\frac{p_0}{d_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

is thus exterior to a circle going through the pole furthest from the origin
- Thus the ROC is given by $|z|>\max_k|\lambda_k|$

- Example - Consider the M-point moving-average FIR filter with an impulse response

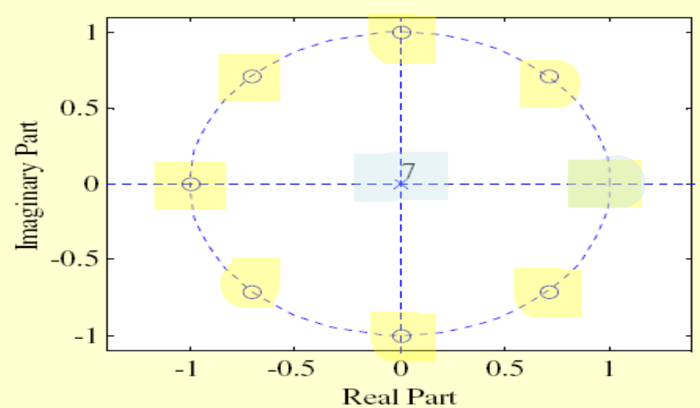
$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^M - 1}{M[z^M(z-1)]}$$

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \le k \le M-1$
- There are $M-1$ poles at $z = 0$ and a single pole at $z = 1$
- The pole at $z = 1$ exactly cancels the zero at $z = 1$
- The ROC is the entire z-plane except $z = 0$

$M = 8$



- Example - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n] = x[n-1] - 1.2\,x[n-2] + x[n-3] + 1.3\,y[n-1] - 1.04\,y[n-2] + 0.222\,y[n-3]$$

- Its transfer function is therefore given by

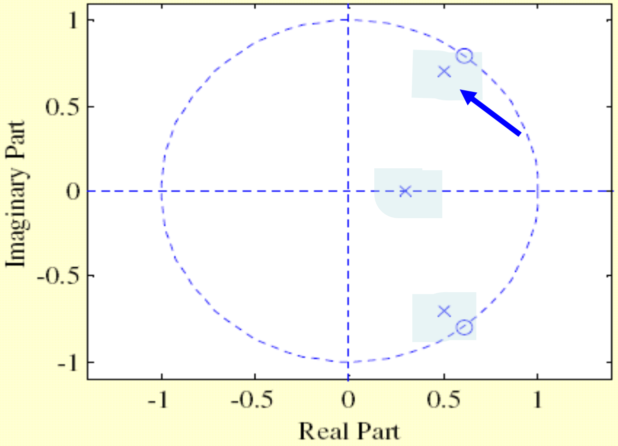
$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$

- Alternate forms:

$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$$

$$= \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$



- ROC: $|z| > \sqrt{0.74}$

Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)}$$

- It is convenient to visualize the contributions of the **zero factor** $(z-\xi_k)$ and the **pole factor** $(z-\lambda_k)$ from the factored form of the frequency response
- The magnitude function is given by

$$|H(e^{j\omega})| = \left|\frac{p_0}{d_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}|e^{j\omega}-\xi_k|}{\prod_{k=1}^{N}|e^{j\omega}-\lambda_k|}$$

which reduces to

$$|H(e^{j\omega})| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}|e^{j\omega}-\xi_k|}{\prod_{k=1}^{N}|e^{j\omega}-\lambda_k|}$$

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M}\arg(e^{j\omega}-\xi_k) - \sum_{k=1}^{N}\arg(e^{j\omega}-\lambda_k)$$

Frequency Response from Transfer Function

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{p_0}{d_0}\right|^2 \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)(e^{-j\omega} - \xi_k^*)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)(e^{-j\omega} - \lambda_k^*)}$$

Geometric Interpretation of Frequency Response Computation

- The factored form of the frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

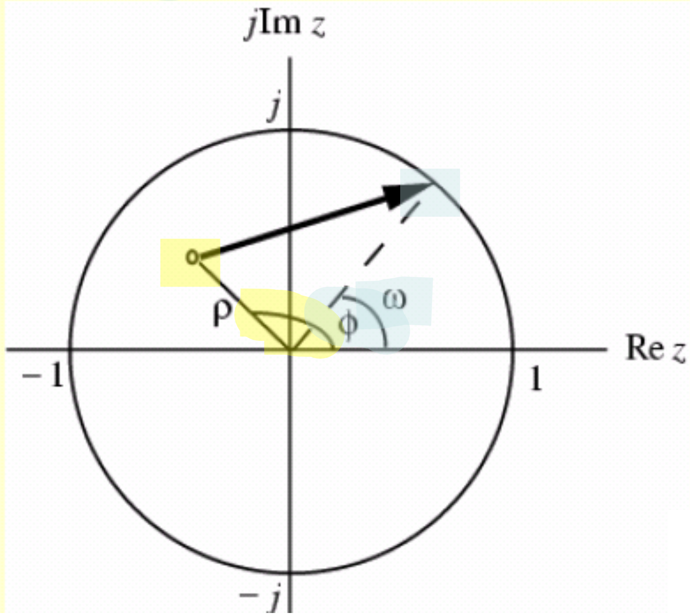
is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency
- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

where $\rho e^{j\phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

- As shown below in the z-plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



- As ω is varied from 0 to 2π, the tip of the vector moves counterclockise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

- As indicated by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

the magnitude response $|H(e^{j\omega})|$ at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M}\arg(e^{j\omega}-\xi_k) - \sum_{k=1}^{N}\arg(e^{j\omega}-\lambda_k)$$

we observe that the phase response at a specific value of ω is obtained by adding the phase of the term p_0/d_0 and the linear-phase term $\omega(N-M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$
  - To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range
  - Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range

Stability Condition in Terms of the Pole Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

- The ROC of the z-transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa

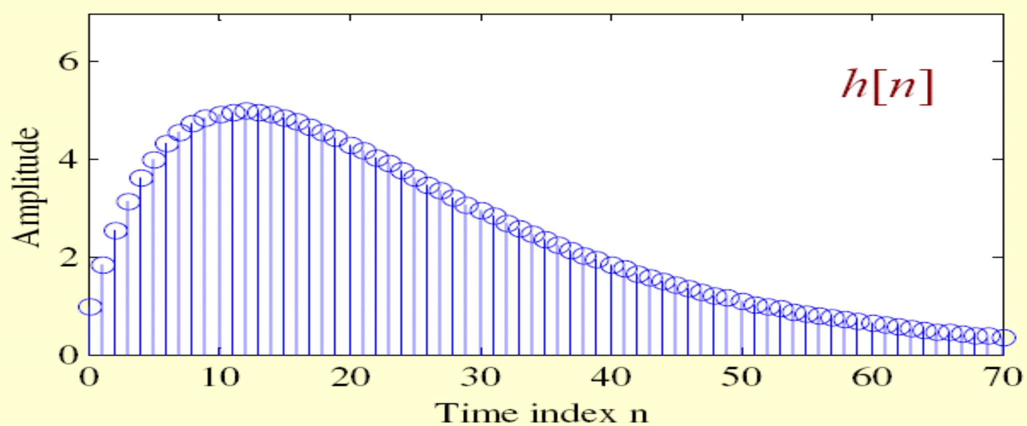
- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z = \infty$
- An FIR digital filter with bounded impulse response is always stable

- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

- Example - Consider the causal IIR transfer function

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

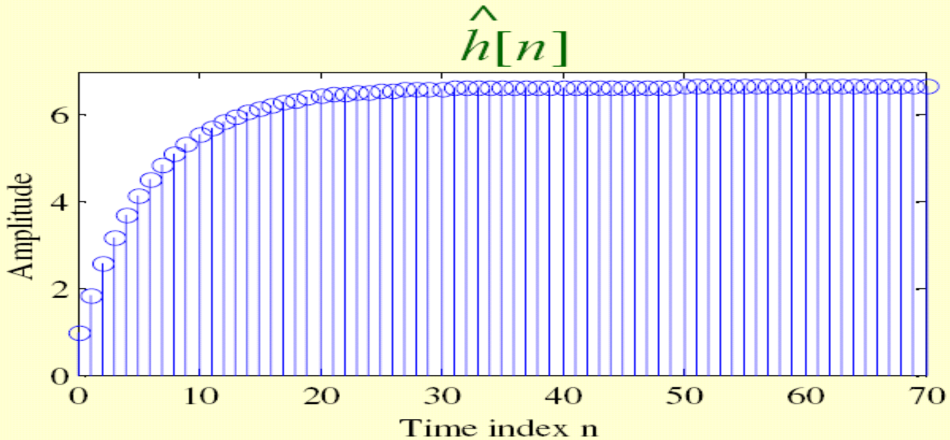


- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases

- The absolute summability condition of $h[n]$ is satisfied
- Hence, $H(z)$ is a stable transfer function
- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

- A plot of the impulse response of $\hat{h}[n]$ is shown below



- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases
- Hence, the absolute summability condition of is violated
- Thus, $\hat{H}(z)$ is an unstable transfer function

- The stability testing of a IIR transfer function is therefore an important problem
- In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- For a causal IIR transfer function, the sum S can be computed approximately as

$$S_K = \sum_{n=0}^{K-1} |h[n]|$$

- The partial sum is computed for increasing values of K until the difference between a series of consecutive values of S_K is smaller than some arbitrarily chosen small number, which is typically 10^{-6}
- For a transfer function of very high order this approach may not be satisfactory
- An alternate, easy-to-test, stability condition is developed next

- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

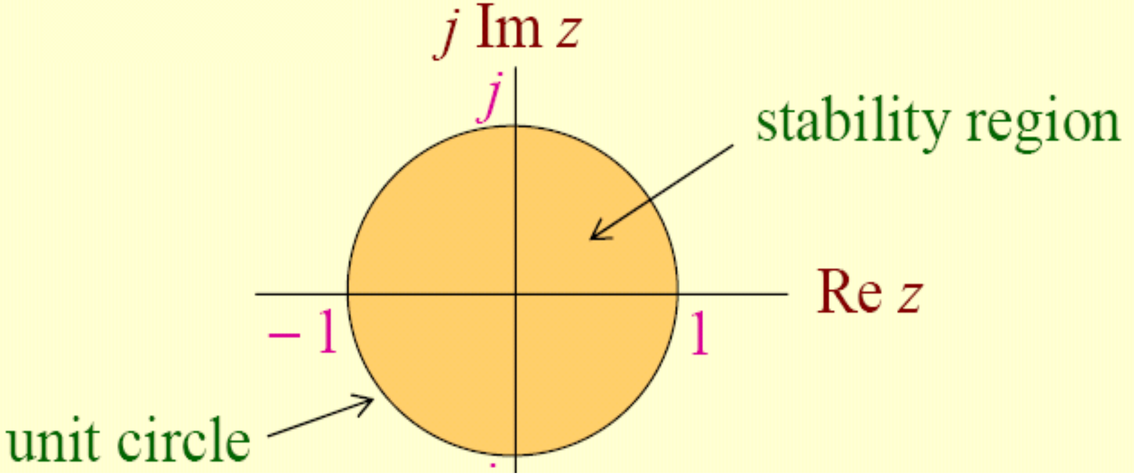
$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- Its impulse response $\{h[n]\}$ is a right-sided sequence
- The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$

- But stability requires that $\{h[n]\}$ be absolutely summable
- This in turn implies that the DTFT $H(e^{j\omega})$ of $\{h[n]\}$ exists
- Now, if the ROC of the *z*-transform $H(z)$ includes the unit circle, then

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle
- The stability region (shown shaded) in the z-plane is shown below



Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$H(z) = \frac{1}{1-0.845z^{-1}+0.850586z^{-2}}$$

is

$$H(z) = \frac{1}{(1-0.902z^{-1})(1-0.943z^{-1})}$$

which has a real pole at $z = 0.902$ and a real pole at $z = 0.943$

- Since both poles are inside the unit circle, $H(z)$ is BIBO stable

Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since both poles are not inside the unit circle, $H(z)$ is unstable