5.15 Problems

5.1 The *periodic convolution* of two periodic sequences, $\tilde{x}[n]$ and $\tilde{h}[n]$, of period N each, is defined by

$$\tilde{y}[n] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{h}[n-r]. \tag{5.184}$$

Show that $\tilde{y}[n]$ is also a periodic sequence of period N.

5.2 Determine the periodic sequence $\tilde{y}[n]$ obtained by a periodic convolution of each pair of periodic sequences of period 5 given below:

(a) $\tilde{x}[n] = \{1 \quad 2 \quad -2 \quad -1 \quad 3\}$, $\tilde{h}[n] = \{2 \quad 0 \quad 1 \quad 3 \quad -4\}$, $0 \leq n \leq 4$,
(b) $\tilde{x}[n] = \{-1 \quad 5 \quad 3 \quad 0 \quad 3\}$, $\tilde{h}[n] = \{-2 \quad 0 \quad 5 \quad 3 \quad -2\}$, $0 \leq n \leq 4$.

5.3 Let $\tilde{x}[n]$ be a periodic sequence with period N, i.e., $\tilde{x}[n] = \tilde{x}[n + \ell N]$, where ℓ is any integer. The sequence $\tilde{x}[n]$ can be represented by a Fourier series given by a weighted sum of periodic complex exponential sequences $\tilde{\Psi}_k[n] = e^{j2\pi kn/N}$. Show that, unlike the Fourier series representation of a periodic continuous-time signal, the Fourier series representation of a periodic discrete-time sequence requires only N of the periodic complex exponential sequences $\tilde{\Psi}_k[n]$, $k = 0, 1, \ldots, N-1$, and is of the form

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j2\pi kn/N}, \tag{5.185a}$$

where the Fourier coefficients $\tilde{X}[k]$ are given by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi kn/N}. \tag{5.185b}$$

Show that $\tilde{X}[k]$ is also a periodic sequence in k with a period N. The set of equations in Eqs. (5.185a) and (5.185b) represent the *discrete Fourier series* pair.

5.4 Determine the discrete Fourier series coefficients, defined in Eq. (5.185b), of the following periodic sequences:
(a) $\tilde{x}_1[n] = \cos(\pi n/4)$, (b) $\tilde{x}_2[n] = \sin(\pi n/3) + 3\cos(\pi n/4)$.

5.5 Show, using Eqs. (5.185a) and (5.185b), that the periodic impulse train $\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n + rN]$ can be expressed in the form $\tilde{p}[n] = \frac{1}{N}\sum_{\ell=0}^{N-1} e^{j2\pi\ell n/N}$.

5.6 Let $x[n]$ be an aperiodic sequence with a DTFT $X(e^{j\omega})$. Define

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = X(e^{j2\pi k/N}), \qquad -\infty < k < \infty .$$

Show that $\tilde{X}[k]$ is a periodic sequence in k with a period N. Let $\tilde{X}[k]$ be the discrete Fourier series coefficients, defined in Eq. (5.185b), of the periodic sequence $\tilde{x}[n]$. Show, using Eqs. (5.185a) and (5.185b), that

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN].$$

5.7 Let $\tilde{x}[n]$ and $\tilde{y}[n]$ be two periodic sequences with period N. Denote their discrete Fourier series coefficients, defined in Eq. (5.185b), as $\tilde{X}[k]$ and $\tilde{Y}[k]$, respectively.

(a) Let $\tilde{g}[n] = \tilde{x}[n]\tilde{y}[n]$ with $\tilde{G}[k]$ denoting its discrete Fourier series coefficients. Show, using Eqs. (5.185a) and (5.185b), that $\tilde{G}[k]$ can be expressed in terms of $\tilde{X}[k]$ and $\tilde{Y}[k]$ as

$$\tilde{G}[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}[\ell]\tilde{Y}[k-\ell]. \tag{5.186}$$

(b) Let $\tilde{H}[k] = \tilde{X}[k]\tilde{Y}[k]$ denote the discrete Fourier series coefficients of a periodic sequence $\tilde{h}[n]$. Show, using Eqs. (5.185a) and (5.185b), that $\tilde{h}[n]$ can be expressed in terms of $\tilde{x}[n]$ and $\tilde{y}[n]$ as

$$\tilde{h}[n] = \sum_{r=0}^{N-1} \tilde{x}[r]\tilde{y}[n-r]. \tag{5.187}$$

5.8 Determine the N-point DFTs of the following length-N sequences defined for $0 \le n \le N-1$:
(a) $x_a[n] = \sin(2\pi n/N)$, (b) $x_b[n] = \cos^2(2\pi n/N)$, (c) $x_c[n] = \cos^3(2\pi n/N)$.

5.9 Determine the N-point DFTs of the following length-N sequences defined for $0 \le n \le N-1$:
(a) $y_a[n] = \alpha^n$, (b) $y_b[n] = \begin{cases} 2, & \text{for } n \text{ even}, \\ -3, & \text{for } n \text{ odd}. \end{cases}$

5.10 Determine the N-point DFT $X[k]$ of the N-point sequence $x[n] = \cos(\omega_o n)$, $0 \le n \le N-1$, for $\omega_o \ne 2\pi r/N$, $0 < r < N-1$.

5.11 Consider a length-N sequence $x[n]$, $0 \le n \le N-1$, with N even. Define 2 subsequences of length-$\frac{N}{2}$ each: $x_0[n] = x[2n]$ and $x_1[n] = x[2n+1]$, $0 \le n \le \frac{N}{2}$. Let $X[k]$, $0 \le k \le N-1$, denote the N-point DFT of $x[n]$, and $X_0[k]$ and $X_1[k]$, $0 \le k \le \frac{N}{2}-1$, denote the $\frac{N}{2}$-point DFTs of $x_0[n]$ and $x_1[n]$, respectively. Express $X[k]$ as a function of $X_0[k]$ and $X_1[k]$.

5.12 Consider a length-N sequence $x[n]$, $0 \le n \le N-1$, with N even. Define two length-$\frac{N}{2}$ sequences given by

$$x_0[n] = \left(x[n] + x[\tfrac{N}{2}+n]\right), \qquad x_1[n] = \left(x[n] - x[\tfrac{N}{2}+n]\right) W_N^n, \qquad 0 \le n \le \tfrac{N}{2}-1.$$

If $X_0[k]$ and $X_1[k]$, $0 \le k \le \frac{N}{2}-1$, denote the $\frac{N}{2}$-point DFTs of $x_0[n]$ and $x_1[n]$, respectively, determine the N-point DFT $X[k]$, $0 \le k \le N-1$, of $x[n]$ from these two $\frac{N}{2}$-point DFTs.

5.13 Let $X[k]$ denote the N-point DFT of a length-N sequence $x[n]$, with N even. Define two length-$\frac{N}{2}$ sequences given by

$$g[n] = \tfrac{1}{2}(x[2n] + x[2n+1]), \quad h[n] = \tfrac{1}{2}(x[2n] - x[2n+1]), \quad 0 \le n \le \tfrac{N}{2}-1.$$

If $G[k]$ and $H[k]$, $0 \le k \le \frac{N}{2}-1$, denote $\frac{N}{2}$-point DFTs of $g[n]$ and $h[n]$, respectively, determine the N-point DFT $X[k]$ from these two $\frac{N}{2}$-point DFTs.

5.14 Let $X[k], 0 \le k \le N-1$, denote the N-point DFT of a length-N sequence $x[n]$, with N even. Define two length-$\frac{N}{2}$ sequences given by

$$g[n] = a_1 x[2n] + a_2 x[2n+1], \quad h[n] = a_3 x[2n] + a_4 x[2n+1], \quad 0 \le n \le \tfrac{N}{2}-1,$$

where $a_1 a_4 \ne a_2 a_3$. If $G[k]$ and $H[k]$, $0 \le k \le \frac{N}{2}-1$, denote $\frac{N}{2}$-point DFTs of $g[n]$ and $h[n]$, respectively, determine the N-point DFT $X[k]$ from these two $\frac{N}{2}$-point DFTs.

5.15 Let $x[n], 0 \leq n \leq N-1$, be a length-N sequence with an N-point DFT given by $X[k], 0 \leq k \leq N-1$. Determine the $2N$-point DFT of each of the following length-$2N$ sequences:

(a) $g[n] = \begin{cases} x[n], & 0 \leq n \leq N-1, \\ 0, & N \leq n \leq 2N-1, \end{cases}$ (b) $h[n] = \begin{cases} 0, & 0 \leq n \leq N-1, \\ x[n-N], & N \leq n \leq 2N-1. \end{cases}$

5.16 Let $G[k]$ and $H[k], 0 \leq k \leq 2N-1$, denote, respectively, the $2N$-point DFTs of the length-$2N$ sequences $g[n]$ and $h[n]$ of Problem 5.15. Define a new length-$2N$ sequence given by $y[n] = g[n] + h[n]$, with a $2N$-point DFT $Y[k], 0 \leq k \leq 2N-1$. Develop the relation between $Y[k]$, $G[k]$, $H[k]$, and $X[k]$.

5.17 Let $Y[k]$ denote the MN-point DFT of a length-N sequence $x[n]$ appended with $(M-1)N$ zeros. Show that the N-point DFT $X[k]$ can be simply obtained from $Y[k]$ as follows:

$$X[k] = Y[kM], \qquad 0 \leq k \leq N-1.$$

5.18 Let $x[n], 0 \leq n \leq N-1$, be a length-N sequence with an N-point DFT given by $X[k], 0 \leq k \leq N-1$. Assume N is odd. Let $R = LN$, where L is a positive integer. Define an R-point DFT $Y[k], 0 \leq k \leq R-1$, given by

$$Y[k] = \begin{cases} LX[k], & 0 \leq k \leq \frac{N-1}{2}, \\ LX[k-R+N], & R - \frac{N-1}{2} \leq k \leq R-1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the length-R IDFT $y[n], 0 \leq n \leq R-1$, of $Y[k]$ as a function of $x[n]$.

5.19 Let $x[n],\ 0 \leq n \leq N-1$, be a length-N sequence with an N-point DFT $X[k],\ 0 \leq k \leq N-1$.

(a) If $x[n]$ is a symmetric sequence satisfying the condition $x[n] = x[\langle N-1-n\rangle_N]$, show that $X[N/2] = 0$ for N even.

(b) If $x[n]$ is a antisymmetric sequence satisfying the condition $x[n] = -x[\langle N-1-n\rangle_N]$, show that $X[0] = 0$.

(c) If $x[n]$ is a sequence satisfying the condition $x[n] = -x[\langle n+M\rangle_N]$ with $N = 2M$, show that $X[2\ell] = 0$ for $\ell = 0, 1, \ldots, M-1$.

5.20 Let $x[n],\ 0 \leq n \leq N-1$, be an even-length sequence with an N-point DFT $X[k],\ 0 \leq k \leq N-1$. If $X[2m] = 0$ for $0 \leq m \leq \frac{N}{2} - 1$, show that $x[n] = -x[\langle n + \frac{N}{2}\rangle_N]$.

5.21 Let $x[n],\ 0 \leq n \leq N-1$, be a length-N sequence with an N-point DFT $X[k],\ 0 \leq k \leq N-1$. Determine the N-point DFTs of the following length-N sequences in terms of $X[k]$:

(a) $w[n] = \alpha x[\langle n - m_1\rangle_N] + \beta x[\langle n - m_2\rangle_N]$, where m_1 and m_2 are positive integers less than N,

(b) $g[n] = \begin{cases} x[n], & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases}$

(c) $y[n] = x[n] \circledN x[n]$.

5.22 Let $x[n],\ 0 \leq n \leq N-1$, be an even-length sequence with an N-point DFT $X[k],\ 0 \leq k \leq N-1$. Determine the N-point DFTs of the following length-N sequences in terms of $X[k]$:

(a) $u[n] = x[n] - x[\langle n - \frac{N}{2}\rangle_N]$, (b) $v[n] = x[n] + x[\langle n - \frac{N}{2}\rangle_N]$, (c) $y[n] = (-1)^n x[n]$.

5.23 Let $x[n],\ 0 \leq n \leq N-1$, be a length-N sequence with an N-point DFT $X[k],\ 0 \leq k \leq N-1$. Determine the N-point inverse DFTs of the following length-N DFTs in terms of $x[n]$:

(a) $W[k] = \alpha X[\langle k - m_1\rangle_N] + \beta X[\langle k - m_2\rangle_N]$, where m_1 and m_2 are positive integers less than N,

(b) $G[k] = \begin{cases} X[k], & \text{for } k \text{ even}, \\ 0, & \text{for } k \text{ odd}, \end{cases}$

(c) $Y[k] = X[k] \circledN X[k]$.

5.24 Let $x[n]$, $0 \le n \le N-1$, be a length-N sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$.

(a) Show that if N is even and if $x[n] = -x[\langle n + \frac{N}{2}\rangle_N]$ for all n, then $X[k] = 0$ for k even.

(b) Show that if N is an integer multiple of 4 and if $x[n] = -x[\langle n + \frac{N}{4}\rangle_N]$ for all n, then $X[k] = 0$ for $k = 4\ell$, $0 \le \ell \le \frac{N}{4} - 1$.

5.25 Let $x[n]$, $0 \le n \le N-1$, be a length-N real sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$.

(a) Show that $X[\langle N-k\rangle_N] = X^*[k]$.

(b) Show that $X[0]$ is real.

(c) If N is even, show that $X[N/2]$ is real.

5.26 Let $x[n]$ be a length-N complex sequence with an N-point DFT $X[k]$. Determine the N-point DFTs of the following length-N sequences in terms of $X[k]$:
(a) $x^*[\langle -n\rangle_N]$, (b) $x_{\rm re}[n]$, (c) $jx_{\rm im}[n]$, (d) $x_{\rm cs}[n]$, (e) $x_{\rm ca}[n]$.

5.27 Let $x[n]$ be a length-N real sequence with an N-point DFT $X[k]$. Prove the following symmetry properties of $X[k]$:
(a) $X[k] = X^*[\langle -k\rangle_N]$, (b) $X_{\rm re}[k] = X_{\rm re}[\langle -k\rangle_N]$, (c) $X_{\rm im}[k] = -X_{\rm im}[\langle -k\rangle_N]$, (d) $|X[k]| = |X[\langle -k\rangle_N]|$, (e) $\arg X[k] = -\arg X[\langle -k\rangle_N]$.

5.28 Without computing the DFT, determine which one of the following length-9 sequences defined for $0 \le n \le 8$ has a real-valued 9-point DFT and which one has an imaginary-valued 9-point DFT.

(a) $\{x_1[n]\} = \{4 \quad 3 \quad -5 \quad 1 \quad -2 \quad -2 \quad 1 \quad -5 \quad 3\}$,

(b) $\{x_2[n]\} = \{0 \quad 5 \quad 1 \quad 4 \quad -3 \quad 3 \quad -4 \quad -1 \quad -5\}$,

(c) $\{x_3[n]\} = \{0 \quad -5 \quad 2 \quad 4 \quad -3 \quad 3 \quad -4 \quad -1 \quad -5\}$,

(d) $\{x_4[n]\} = \{-5 \quad 5 \quad -2 \quad 2 \quad 4 \quad 4 \quad 2 \quad -2 \quad 5\}$.

5.29 Let $G[k]$ and $H[k]$, $0 \le k \le 7$, denote the 8-point DFTs of two length-8 sequences, $g[n]$ and $h[n]$, $0 \le n \le 7$, respectively.

(a) If $G[k] = \{2.6 + j4.1 \quad 3 - j2.7 \quad -4.2 + j1.4 \quad 3.5 - j2.6 \quad 0.5 \quad 1.3 + j4.4 \quad 2.4 - j1.6 \quad -3 + j1.6\}$ and $h[n] = g[\langle n-5\rangle_8]$, determine $H[k]$ without forming $h[n]$ and then computing its DFT.

(b) If $g[n] = \{-0.1 - j0.7 \quad 1.3 + j \quad 2 + j0.7 \quad 1.1 + j2.2 \quad -0.8 + j0.2 \quad 3.4 - j0.1 \quad -1.2 + j3.1 \quad j1.5\}$ and $H[k] = G[\langle k+3\rangle_8]$, determine $h[n]$ without computing the DFT $G[k]$, forming $H[k]$, and then finding its inverse DFT.

5.30 Prove the following general properties of the DFT listed in Table 5.3: (a) linearity, (b) circular time-shifting, (c) circular frequency-shifting, (d) duality, and (e) N-point circular convolution.

5.31 Prove Eq. (5.116).

5.32 Consider two length-N real-valued sequences $x[n]$ and $y[n]$ defined for $0 \le n \le N-1$ with N-point DFTs $X[k]$ and $Y[k]$, $0 \le k \le N-1$, respectively. The *circular correlation* of $x[n]$ and $y[n]$ is given by

$$r_{xy}[\ell] = \sum_{n=0}^{N-1} x[n]y[\langle \ell + n\rangle_N], \quad 0 \le \ell \le N-1. \tag{5.188}$$

Express the DFT of $r_{xy}[\ell]$ in terms $X[k]$ and $Y[k]$.

5.33 Let $x[n]$, $0 \leq n \leq N-1$, be a length-N sequence with an MN-point DFT $X[k]$, $0 \leq k \leq MN-1$. Define

$$y[n] = x[\langle n \rangle_N], \qquad 0 \leq n \leq MN-1.$$

How would you compute the MN-point DFT $Y[k]$ of $y[n]$ knowing only $X[k]$?

5.34 Consider the length-10 sequence, defined for $0 \leq n \leq 9$,

$$\{x[n]\} = \{-3 \quad 5 \quad 45 \quad -15 \quad -9 \quad -19 \quad -8 \quad 21 \quad -10 \quad 23\},$$

with a 10-point DFT given by $X[k]$, $0 \leq k \leq 9$. Evaluate the following functions of $X[k]$ without computing the DFT:

(a) $X[0]$, (b) $X[5]$, (c) $\sum_{k=0}^{9} X[k]$, (d) $\sum_{k=0}^{9} e^{-j2\pi k/5} X[k]$, (e) $\sum_{k=0}^{9} |X[k]|^2$.

5.35 Let $X[k]$, $0 \leq k \leq 11$, be a 12-point DFT of a length-12 real sequence $x[n]$ with first 7 samples of $X[k]$ given by

$$X[k] = \{11 \quad 8-j2 \quad 1-j12 \quad 6+j3 \quad -3+j2 \quad 2+j \quad 15\}, \; 0 \leq k \leq 6.$$

Determine the remaining samples of $X[k]$. Evaluate the following functions of $x[n]$ without computing the IDFT of $X[k]$:

(a) $x[0]$, (b) $x[6]$, (c) $\sum_{n=0}^{11} x[n]$, (d) $\sum_{n=0}^{11} e^{j2\pi n/3} x[n]$, (e) $\sum_{n=0}^{11} |x[n]|^2$.

5.36 Let $g[n]$ and $h[n]$ be two finite-length sequences of length 7 each. If $y_L[n]$ and $y_C[n]$ denote the linear and 7-point circular convolutions of $g[n]$ and $h[n]$, respectively, express $y_C[n]$ in terms of $y_L[n]$.

5.37 The even samples of the 9-point DFT of a length-9 real sequence are given by $X[0] = -5.7$, $X[2] = 1.2 - j4.1$, $X[4] = -3.5 + j5.3$, $X[6] = 8.6 - j9.6$, and $X[8] = -7.7 - j3.2$. Determine the missing odd samples of the DFT.

5.38 The following 5 samples of the 9-point DFT $X[k]$ of a real length-9 sequence are given: $X[0] = 11$, $X[2] = 1.2 - j2.3$, $X[3] = -7.2 - j4.1$, $X[5] = -3.1 + j8.2$, and $X[8] = 4.5 + j1.6$. Determine the remaining 4 samples.

5.39 The following 7 samples of a length-12 real sequence $x[n]$ with a real-valued 12-point DFT $X[k]$ are given by $x[0] = 3.8$, $x[2] = 0.7$, $x[3] = -3.25$, $x[5] = 4.1$, $x[6] = 2.87$, $x[8] = 9.3$, and $x[11] = -2$. Find the remaining 5 samples of $x[n]$.

5.40 The first 7 samples of a length-12 real sequence $x[n]$ with an imaginary-valued 12-point DFT $X[k]$ are given by $x[0] = 0$, $x[1] = 0.7$, $x[2] = -3.25$, $x[3] = 4.1$, $x[4] = 2.87$, $x[5] = -9.3$, and $x[6] = 0$. Find the remaining 5 samples of $x[n]$.

5.41 A 174-point DFT $X[k]$ of a real-valued sequence $x[n]$ has the following DFT samples: $X[0] = 11$, $X[9] = -3.4 + j5.9$, $X[k_1] = 7.1 + j2.4$, $X[51] = 5 - j1.6$, $X[k_2] = 8.7 + j4.9$, $X[87] = 4.5$, $X[113] = 8.7 - j4.9$, $X[k_3] = 5 + j1.6$, $X[162] = 7.1 - j2.4$, and $X[k_4] = -3.4 - j5.9$. Remaining DFT samples are assumed to be of zero value.

(a) Determine the values of the indices k_1, k_2, k_3, and k_4.
(b) What is the dc value of $\{x[n]\}$?
(c) Determine the expression for $\{x[n]\}$ without computing the IDFT.
(d) What is the energy of $\{x[n]\}$?

5.42 A 126-point DFT $X[k]$ of a real-valued sequence $x[n]$ has the following DFT samples: $X[0] = 12.8 + j\alpha$, $X[13] = -3.7 + j2.2$, $X[k_1] = 9.1 - j5.4$, $X[k_2] = 6.3 + j2.3$, $X[51] = -j1.7$, $X[63] = 13 + j\beta$, $X[k_3] = \gamma + j1.7$, $X[79] = 6.3 + j\delta$, $X[108] = \epsilon + j5.4$, and $X[k_4] = -3.7 - j2.2$. Remaining DFT samples are assumed to be of zero value.

(a) Determine the values of the indices k_1, k_2, k_3, and k_4.
(b) Determine the values of α, β, δ, and ϵ.
(c) What is the dc value of $\{x[n]\}$?
(d) Determine the expression for $\{x[n]\}$ without computing the IDFT.
(e) What is the energy of $\{x[n]\}$?

5.43 A length-9 sequence is given by $\{x[n]\} = \{3,\ 5,\ 1,\ 4,\ -3,\ 5,\ -2,\ -2,\ 4\}$, $0 \le n \le 8$, with an 9-point DFT given by $X[k]$, $0 \le k \le 8$. Without computing the IDFT, determine the sequence $y[n]$ whose 9-point DFT is given by $Y[k] = W_3^{-2k} X[k]$.

5.44 The first 5 samples of the 9-point DFT $H[k]$, $0 \le k \le 8$, of a length-9 real sequence $h[n]$, $0 \le n \le 8$, are given by

$$H[k] = \{15 \quad 6.8414 - j6.0572 \quad 6.0346 - j1.957 \quad j8.6603 \quad -6.876 - j11.4883\}.$$

Determine the 9-point DFT $G[k]$ of the length-9 sequence $e^{j2\pi n/3}h[n]$ without computing $h[n]$, forming the sequence $g[n]$, and then taking its DFT.

5.45 Consider the two finite-length sequences $g[n] = \{2 \ \ -1 \ \ 3\}, 0 \le n \le 2$ and $h[n] = \{-2 \ \ 4 \ \ 2 \ \ -1\}, 0 \le n \le 3$.

(a) Determine $y_L[n] = g[n] \circledast h[n]$.
(b) Extend $g[n]$ to a length-4 sequence $g_e[n]$ by zero-padding and compute $y_C[n] = g_e[n] \textcircled{4} h[n]$.
(c) Determine $y_C[n]$ using the DFT-based approach.
(d) Extend $g[n]$ and $h[n]$ to length-6 sequences by zero-padding and compute the 6-point circular convolution $y[n]$ of the extended sequences. Is $y[n]$ the same as $y_L[n]$ determined in Part (a)?

5.46 Show that the circular convolution is commutative.

5.47 Consider two length-N sequences $x_1[n]$ and $x_2[n]$ defined for $0 \le n \le N-1$. Let $y[n] = x_1[n] \textcircled{N} x_2[n]$. Prove the following equalities:

(a) $$\sum_{n=0}^{N-1} y[n] = \left(\sum_{n=0}^{N-1} x_1[n]\right)\left(\sum_{n=0}^{N-1} x_2[n]\right),$$

(b) $$\sum_{n=0}^{N-1} (-1)^n y[n] = \left(\sum_{n=0}^{N-1} (-1)^n x_1[n]\right)\left(\sum_{n=0}^{N-1} (-1)^n x_2[n]\right) \quad \text{for } N \text{ even.}$$

5.48 Let $x[n]$ be a length-N sequence with an N-point DFT given by $X[k]$. Assume N is divisible by 3. Define a sequence

$$y[n] = x[3n], \qquad 0 \le n \le \tfrac{N}{3} - 1.$$

Express the $\frac{N}{3}$-point DFT $Y[k]$ of $y[n]$ in terms of $X[k]$.

5.49 The 8-point DFT of a length-8 complex sequence $v[n] = x[n] + jy[n]$ is given by

$$V[0] = 3 + j7, \quad V[1] = -2 + j6, \quad V[2] = 1 - j5, \quad V[3] = 4 - j9,$$

$$V[4] = 5 + j2, \quad V[5] = 3 - j2, \quad V[6] = j4, \quad V[7] = -3 - j8,$$

where $x[n]$ and $y[n]$ are, respectively, the real and imaginary parts of $v[n]$. Without computing the IDFT of $V[k]$, determine the 8-point DFTs $X[k]$ and $Y[k]$ of the real sequences $x[n]$ and $y[n]$, respectively. Verify your result by computing the IDFT of $V[k]$ using MATLAB.

5.50 Determine the 4-point DFTs of each sequence of the following pairs of length-4 sequences defined for $0 \le n \le 3$ by computing a single DFT:

(a) $g[n] = \{2 \quad -1 \quad 3 \quad 0\}$, $h[n] = \{-2 \quad 4 \quad 2 \quad -1\}$,
(b) $x[n] = \{-3 \quad -2 \quad 2 \quad 4\}$, $y[n] = \{1 \quad 2 \quad 3 \quad 4\}$.

HW **5.51** Consider a rational discrete-time Fourier transform $X(e^{j\omega})$ with real coefficients of the form of

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{D(e^{j\omega})} = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_{M-1} e^{-j\omega(M-1)}}{d_0 + d_1 e^{-j\omega} + \cdots + d_{N-1} e^{-j\omega(N-1)}}.$$

Let $P[k]$ denote the M-point DFT of the numerator coefficients $\{p_i\}$ and $D[k]$ denote the N-point DFT of the denominator coefficients $\{d_i\}$. Determine the exact expressions of the DTFT $X(e^{j\omega})$ for $M = N = 4$ if the 4-point DFTs of its numerator and denominator coefficients are given by

$$P[k] = \{-5,\ -2 + j5,\ 4,\ -2 - j5\}, \qquad D[k] = \{3,\ 4 + j,\ -7,\ 4 - j\}.$$

Verify your result using MATLAB.

HW **5.52** Repeat Problem 5.51 for the 4-point DFTs of the numerator and denominator coefficients given by

$$P[k] = \{8,\ -5 - j6,\ -3,\ -5 + j6\}, \qquad D[k] = \{-6,\ 6 + j2,\ 5,\ 6 - j2\}.$$

5.53 Consider a length-N sequence $x[n]$ with a DTFT $X(e^{j\omega})$. Define an M-point DFT $\hat{X}[k] = X(e^{j\omega_k})$, where $\omega_k = 2\pi k/M, k = 0, 1, \ldots, M-1$. Denote the inverse DFT of $\hat{X}[k]$ as $\hat{x}[n]$, which is a length-M sequence. Express $x[n]$ in terms of $\hat{x}[n]$ and show that $x[n]$ can be fully recovered from $\hat{x}[n]$ only if $M \ge N$.

5.54 Let $X(e^{j\omega})$ denote the DTFT of the length-9 sequence $x[n] = \{1 \quad -2 \quad 3 \quad -4 \quad 5 \quad -4 \quad 3 \quad -2 \quad 1\}$.

(a) For the DFT sequence $X_1[k]$, obtained by sampling $X(e^{j\omega})$ at uniform intervals of $\pi/6$ starting from $\omega = 0$, determine the IDFT $x_1[n]$ of $X_1[k]$ without computing $X(e^{j\omega})$ and $X_1[k]$. Can you recover $x[n]$ from $x_1[n]$?

(b) For the DFT sequence $X_2[k]$, obtained by sampling $X(e^{j\omega})$ at uniform intervals of $\pi/4$ starting from $\omega = 0$, determine the IDFT $x_2[n]$ of $X_2[k]$ without computing $X(e^{j\omega})$ and $X_2[k]$. Can you recover $x[n]$ from $x_2[n]$?

5.55 Let $x[n]$ be a length-N sequence with $X[k]$ denoting its N-point DFT. We represent the DFT operation as $X[k] = \mathcal{F}\{x[n]\}$. Determine the sequence $y[n]$ obtained by applying the DFT operation 4 times to $x[n]$, i.e.,

$$y[n] = \mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{x[n]\}\}\}\}.$$

5.56 Let $x[n]$ and $h[n]$ be two length-51 sequences defined for $0 \le n \le 50$. It is known that $h[n] = 0$ for $0 \le n \le 16$ and $37 \le n \le 50$. Denote the 51-point circular convolution of these two sequences as $u[n]$ and their linear convolution as $y[n]$. Determine the range of n for which $y[n] = u[n]$.

5.57 The linear convolution of a length-110 sequence with a length-1300 sequence is to be computed using 128-point DFTs and IDFTs.

(a) Determine the smallest number of DFTs and IDFTs needed to compute the above linear convolution using the overlap-add approach.

(b) Determine the smallest number of DFTs and IDFTs needed to compute the above linear convolution using the overlap-save approach.

5.58 (a) Consider a length-N sequence $x[n]$, $0 \le n \le N-1$, with an N-point DFT $X[k]$, $0 \le k \le N-1$. Define a sequence $y[n]$ of length LN, $0 \le n \le NL - 1$, given by

$$y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \ldots, (N-1)L, \\ 0, & \text{otherwise,} \end{cases} \tag{5.189}$$

where L is a positive integer. Express the NL-point DFT $Y[k]$ of $y[n]$ in terms of $X[k]$.

(b) The 5-point DFT $X[k]$ of a length-5 sequence $x[n]$ is shown in Figure P5.1. Sketch the 21-point DFT $Y[k]$ of a length-21 sequence $y[n]$ generated using Eq. (5.189).

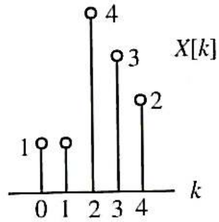

Figure P5.1

5.59 Consider two real, symmetric length-N sequences, $x[n]$ and $y[n]$, $0 \le n \le N-1$ with N even. Define the length-$\frac{N}{2}$ sequences:

$$x_0[n] = x[2n+1] + x[2n], \quad x_1[n] = x[2n+1] - x[2n],$$
$$y_0[n] = y[2n+1] + y[2n], \quad y_1[n] = y[2n+1] - y[2n],$$

where $0 \le n \le \frac{N}{2} - 1$. It can be easily shown that $x_0[n]$ and $y_0[n]$ are real, symmetric sequences of length-$\frac{N}{2}$ each. Likewise, the sequences $x_1[n]$ and $y_1[n]$ are real and antisymmetric sequences. Denote the $\frac{N}{2}$-point DFTs of $x_0[n]$, $x_1[n]$, $y_0[n]$, and $y_1[n]$ by $X_0[k]$, $X_1[k]$, $Y_0[k]$, and $Y_1[k]$, respectively. Define a length-$\frac{N}{2}$ sequence $u[n]$:

$$u[n] = x_0[n] + y_1[n] + j(x_1[n] + y_0[n]).$$

Determine $X_0[k]$, $X_1[k]$, $Y_0[k]$, and $Y_1[k]$ in terms of the $\frac{N}{2}$-point DFT $U[k]$ of $u[n]$.

5.60 The *generalized discrete Fourier transform* (GDFT) is a generalization of the conventional DFT to allow shifts in either or both indices of the transform kernel [Bon76]. The N-point generalized discrete Fourier transform $X_{\text{GDFT}}[k, a, b]$ of a length-N sequence $x[n]$ is defined by

$$X_{\text{GDFT}}[k, a, b] = \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi(n+a)(k+b)}{N}\right). \tag{5.190}$$

Show that the inverse GDFT is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_{\text{GDFT}}[k, a, b] \exp\left(j\frac{2\pi(n+a)(k+b)}{N}\right). \tag{5.191}$$

5.61 Prove the following properties of the DCT: (a) linearity property of the DCT, given by Eq. (5.154), (b) symmetry property of the DCT, given by Eq. (5.155), and (c) energy preservation property, given by Eq. (5.156).

5.62 The N coefficients of the normalized DCT $X^{(n)}_{\text{DCT}}[k]$, $0 \le k \le N-1$, given by Eq. (5.158), can be written in a matrix form $\mathbf{X}_{\text{DCT}} = \mathbf{C}_N \mathbf{x}$, where

$$\mathbf{X}_{\text{DCT}} = \left[X^{(n)}_{\text{DCT}}[0] \quad X^{(n)}_{\text{DCT}}[1] \quad \cdots \quad X^{(n)}_{\text{DCT}}[N-1]\right]^t, \ \mathbf{x} = [x[0] \quad x[1] \quad \cdots \quad x[N-1]]^t,$$

and $\mathbf{C}_N$ is the $N \times N$ DCT matrix whose (k, n)th element is given by

$$X^{(n)}_{\text{DCT}}[k] = \sqrt{\frac{2}{N}}\beta[k]\sum_{n=0}^{N-1}\cos\left(\frac{\pi k(2n+1)}{2N}\right),$$

with $\beta[k]$ given by Eq. (5.160). Even though the DCT matrix $\mathbf{C}_N$ is orthogonal, i.e., $\mathbf{x} = \mathbf{C}_N^{-1}\mathbf{X}_{\text{DCT}} = \mathbf{C}_N^t\mathbf{X}_{\text{DCT}}$, its elements are irrational numbers and do not produce the original input vector $\mathbf{x}$ by applying the inverse DCT transformation to $\mathbf{X}_{\text{DCT}}$ when implemented with finite-precision arithmetic. It is thus desirable in practice to make use of integer-valued orthogonal transform matrix with a uniform frequency decomposition similar to that of the DCT.

(a) One such transform proposed for the H.26L video compression standard is the 4×4 matrix [Bjo98]:

$$\mathbf{H}_N = \begin{bmatrix} 13 & 13 & 13 & 13 \\ 17 & 7 & -7 & -17 \\ 13 & -13 & -13 & 13 \\ 7 & -17 & 17 & -7 \end{bmatrix}.$$

Show that the above transform matrix is orthogonal and all its rows have the same $\mathcal{L}_2$ norm.

(b) A recently proposed simpler 4×4 transform matrix [Mal2002]:

$$\mathbf{G}_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix},$$

has a much smaller dynamic range than $\mathbf{H}_N$. Show that the rows of the above transform matrix are orthogonal but do not have the same $\mathcal{L}_2$ norm.

5.63 Prove the following properties of the Haar transform: (a) orthogonality property, given by Eq. (5.171) and (b) energy conservation property, given by Eq. (5.174).

5.64 The N-point *discrete Hartley transform* (DHT) $X_{\text{DHT}}[k]$ of a length-N sequence $x[n]$ is defined by [Bra83]

$$X_{\text{DHT}}[k] = \sum_{n=0}^{N-1} x[n] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right), \quad k = 0, 1, \ldots, N-1. \tag{5.192}$$

As can be seen from the above, the DHT of a real sequence is also a real sequence. Show that the inverse discrete Hartley transform (DHT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DHT}}[k] \left(\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right), \quad n = 0, 1, \ldots, N-1. \tag{5.193}$$

5.65 Let $X_{\text{DHT}}[k]$ denote the N-point DHT of a length-N sequence $x[n]$.

(a) Show that the DHT of $x[\langle n - n_0 \rangle_N]$ is given by

$$X_{\text{DHT}}[k] \cos\left(\frac{2\pi n_0 k}{N}\right) + X_{\text{DHT}}[-k] \sin\left(\frac{2\pi n_0 k}{N}\right).$$

(b) Determine the N-point DHT of $x[\langle -n \rangle_N]$.

(c) Prove the Parseval's relation:

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DHT}}^2[k]. \tag{5.194}$$

5.66 Develop the relation between the N-point DHT $X_{\text{DHT}}[k]$ and the N-point DFT $X[k]$ of a length-N sequence $x[n]$.

5.67 Let the N-point DHTs of the three length-N sequences $x[n]$, $g[n]$, and $y[n]$ be denoted by $X_{\text{DHT}}[k]$, $G_{\text{DHT}}[k]$, and $Y_{\text{DHT}}[k]$, respectively. If $y[n] = x[n] \circledN g[n]$, show that

$$\begin{aligned} Y_{\text{DHT}}[k] = {} & \tfrac{1}{2} X_{\text{DHT}}[k](G_{\text{DHT}}[k] + G_{\text{DHT}}[\langle -k \rangle_N]) \\ & + \tfrac{1}{2} X_{\text{DHT}}[\langle -k \rangle_N](G_{\text{DHT}}[k] - G_{\text{DHT}}[\langle -k \rangle_N]). \end{aligned} \tag{5.195}$$

5.68 The *discrete combined Fourier transform* (DCFT) of a length-N sequence $x[n]$, $0 \le n \le N-1$, is defined as a linear combination of its N-point DFT and the N-point IDFT given by [Ans85]

$$X_{\text{DCFT}}[k] = \sum_{n=0}^{N-1} \left(\alpha_1 W_N^{nk} + \alpha_2 W_N^{-nk}\right) x[n], \quad 0 \le k \le N-1, \tag{5.196}$$

where at least one of the constants α_1 and α_2 is nonzero.

(a) Consider the sequence

$$y[n] = \sum_{n=0}^{N-1} \left(\beta_1 W_N^{-nk} + \beta_2 W_N^{nk}\right) X_{\text{DCFT}}[k], \quad 0 \le n \le N-1. \tag{5.197}$$

Show that $y[n] = x[n]$, the inverse DCFT of $X_{\text{DCFT}}[k]$ if the following two conditions are satisfied:

$$\alpha_2\beta_1 + \alpha_1\beta_2 = 0,$$
$$N(\alpha_1\beta_1 + \alpha_2\beta_2) = 1.$$

(b) If $\alpha_2^2 \neq \alpha_2^2$, then show that the inverse DCFT of $X_{\text{DCFT}}[k]$ can be expressed as

$$x[n] = \frac{1}{N\left(\alpha_1^2 - \alpha_2^2\right)} \sum_{k=0}^{N-1} \left(\alpha_1 W_N^{-nk} - \alpha_2 W_N^{nk}\right) X_{\text{DCFT}}[k], \quad 0 \le n \le N-1. \tag{5.198}$$

(c) Show that $X_{\text{DCFT}}[k]$ is a real sequence if $\alpha_1 = \alpha_2^* = \alpha_{\text{re}} + j\alpha_{\text{im}}$, provided $\alpha_{\text{re}} \neq 0$, and $\alpha_{\text{im}} \neq 0$.
(d) Show that the discrete Hartley transform is a special case of the real-valued DCFT.

5.69 The Hadamard transform $X_{\text{HT}}[k]$ of a length-N sequence $x[n]$, $n = 0, 1, \ldots, N-1$, is given by [Gon2002]

$$X_{\text{HT}}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n](-1)^{\sum_{i=0}^{\ell-1} b_i(n) b_i(k)}, \qquad k = 0, 1, \ldots, N-1, \tag{5.199}$$

where $b_i(r)$ is the ith bit in the binary representation of r, and $N = 2^\ell$. In matrix form, the Hadamard transform can be represented as

$$\mathbf{X}_{\text{HT}} = \mathbf{H}_N \mathbf{x},$$

where

$$\mathbf{X}_{\text{HT}} = [X_{\text{HT}}[0] \quad X_{\text{HT}}[1] \quad \cdots \quad X_{\text{HT}}[N-1]]^t,$$
$$\mathbf{x} = [x[0] \quad x[1] \quad \cdots \quad x[N-1]]^t.$$

(a) Determine the form of the Hadamard matrix $\mathbf{H}_N$ for $N = 2, 4$, and 8.
(b) Show that

$$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}, \qquad \mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}.$$

(c) Determine the expression for the inverse Hadamard transform.

5.16 MATLAB Exercises

M 5.1 Using MATLAB, compute the N-point DFTs of the length-N sequences of Problem 3.19 for $N = 4, 6, 8,$ and 10. Compare your results with that obtained by evaluating the DTFTs computed in Problem 3.19 at $\omega = 2\pi k/N$, $k = 0, 1, \ldots, N-1$.

M 5.2 Write a MATLAB program to compute the circular convolution of two length-N sequences via the DFT-based approach. Using this program, determine the circular convolution of the following pairs of sequences:

(a) $g[n] = \{5,\ -2,\ 2,\ 0,\ 4,\ 3\}$, $h[n] = \{3,\ 1,\ -2,\ 2,\ -4,\ 4\}$,

(b) $x[n] = \{2 - j,\ -1 - j3,\ 4 - j3,\ 1 + j2,\ 3 + j2\}$, $v[n] = \{-3,\ 2 + j4,\ -1 + j4,\ 4 + j2,\ -3 + j\}$,

(c) $x[n] = \cos(\pi n/2)$, $y[n] = 3^n$, $0 \le n \le 4$.

Verify your result using the function `circonv`.

circonv.m

M 5.3 Using MATLAB, verify the symmetry relations of the DFT of a complex sequence as listed in Table 5.1.

M 5.4 Using MATLAB, verify the symmetry relations of the DFT of a real sequence as listed in Table 5.2.

M 5.5 Using MATLAB, prove the following general properties of the DFT listed in Table 5.3: (a) linearity, (b) circular time-shifting, (c) circular frequency-shifting, (d) duality, (e) N-point circular convolution, (f) modulation, and (g) Parseval's relation.

M 5.6 Write a MATLAB program to compute the DFTs of two real sequences of equal lengths based on the method outlined in Section 5.9.1.

M 5.7 Verify the results of Problem 5.34 by computing the DFT $X[k]$ of the sequence $x[n]$ given using MATLAB, and then evaluate the functions of $X[k]$ listed.

M 5.8 Verify the results of Problem 5.35 by computing the IDFT $x[n]$ of the DFT $X[k]$ given using MATLAB, and then evaluate the functions of $x[n]$ listed.

M 5.9 Write a MATLAB program to implement the Fourier-domain filtering illustrated in Example 5.13. Using this program, verify the results of this example.

M 5.10 Write a MATLAB function to implement the overlap-save method. Using this function, demonstrate the filtering of the noise-corrupted signal of Example 2.13 using a length-3 moving average filter by modifying Program 3_6.

Program 3_6