Example 6.42 reexamines the stability of the causal LTI system given by Eq. (6.106) of Example 6.40 from the pole locations of the transfer function of the system.

EXAMPLE 6.42 Causal LTI IIR System with Infinite Number of Unit Circle Poles

The transfer function $H(z)$ of the causal LTI discrete-time system of Example 6.40 is given by the z-transform of the impulse response of Eq. (6.106). It is given by

$$H(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} = \log_e\left(\frac{1}{1-z^{-1}}\right), \qquad |z| > 1, \tag{6.107}$$

which has infinite number of poles on the unit circle at $z = 1$, and hence, it is unstable.

On the other hand, an anticausal digital filter has a left-sided impulse response $\{h[n]\}$ and as a result, the ROC of its transfer function $H(z)$ is interior to the circle going through the pole that is closest to the origin. However, for BIBO stability, the Fourier transform $H(e^{j\omega})$ of $\{h[n]\}$ must exist, implying that the unit circle of $H(z)$ lies in its ROC. Hence, in this case, all poles of a stable anticausal transfer function $H(z)$ must be strictly outside the unit circle. This type of filter will thus have a stable response by running time backwards. In practice, such an anticausal transfer function can be implemented by storing a finite length of the output data in a buffer and reading it in a reverse order.

6.8 Summary

The z-transform of an aperiodic sequence has been introduced and its properties reviewed. As in the case of the discrete-time Fourier transform discussed in Chapter 3, and the discrete Fourier transform, the discrete cosine transform, and the Haar transform discussed in Chapter 5, this alternate representation reviewed in this chapter also consists of a pair of expressions: the analysis equation and the synthesis equation. The analysis equation is used to convert from the time-domain representation to the transform-domain representation, while the synthesis equation is used for the reverse process.

An important and useful characterization of an LTI discrete-time system is its transfer function given by the z-transform of its impulse response. The properties of the transfer function have been studied, and the stability condition of an LTI system in terms of the pole locations of its transfer function has been derived.

6.9 Problems

6.1 Show that for a causal sequence $x[n]$ defined for $n \geq 0$ and with a z-transform $X(z)$,

$$x[0] = \lim_{z\to\infty} X(z).$$

The above result is known as the *initial value theorem*.

6.2 Derive the z-transforms and the ROCs of the following sequences given in Table 6.1:
(a) $\delta[n]$, (b) $n\alpha^n\mu[n]$, and (c) $(r^n \sin\omega_0 n)\mu[n]$.

6.3 Determine the z-transforms of the following sequences and their respective ROCs:
(a) $x_1[n] = \alpha^n\mu[n-2]$, (b) $x_2[n] = -\alpha^n\mu[-n-3]$, (c) $x_3[n] = \alpha^n\mu[n+4]$, (d) $x_4[n] = \alpha^n\mu[-n]$.

6.4 Determine which one of the following four sequences has the same z-transform:
(a) $x_1[n] = (0.4)^n\mu[n] + (-0.6)^n\mu[n]$, (b) $x_2[n] = (0.4)^n\mu[n] - (-0.6)^n\mu[-n-1]$,
(c) $x_3[n] = -(0.4)^n\mu[-n-1] - (-0.6)^n\mu[-n-1]$, (d) $x_4[n] = -(0.4)^n\mu[-n-1] + (-0.6)^n\mu[n]$.

6.5 Consider the following sequences:
(i) $x_1[n] = (0.3)^n\mu[n+1]$, (ii) $x_2[n] = (0.7)^n\mu[n-1]$, (iii) $x_3[n] = (0.4)^n\mu[n-5]$,
(iv) $x_4[n] = (-0.4)^n\mu[-n-2]$.
(a) Determine the ROCs of the z-transform of each of the above sequences.
(b) From the ROCs determined in Part (a), determine the ROCs of the following sequences:
(i) $y_1[n] = x_1[n] + x_2[n]$, (ii) $y_2[n] = x_1[n] + x_3[n]$, (iii) $y_3[n] = x_1[n] + x_4[n]$,
(iv) $y_4[n] = x_2[n] + x_3[n]$, (v) $y_5[n] = x_2[n] + x_4[n]$, (vi) $y_6[n] = x_3[n] + x_4[n]$.

6.6 Determine the z-transform of the two-sided sequence $v[n] = \alpha^{|n|}$, $|\alpha| < 1$. What is its ROC?

6.7 Determine the z-transform of each of the following sequences and their respective ROCs. Assume $|\beta| > |\alpha| > 0$. Show their pole-zero plots and indicate clearly the ROC in these plots.
(a) $x_1[n] = (\alpha^n + \beta^n)\mu[n+2]$, (b) $x_2[n] = \alpha^n\mu[-n-2] + \beta^n\mu[n-1]$,
(c) $x_3[n] = \alpha^n\mu[n+1] + \beta^n\mu[-n-2]$.

6.8 Consider the z-transform

$$G(z) = \frac{(z^2 + 0.2z + 0.1)(z^2 - z + 0.5)}{(z^2 + 0.3z - 0.18)(z^2 - 2z + 4)}. \tag{6.108}$$

There are four possible nonoverlapping regions of convergence (ROCs) of this z-transform. Discuss the type of inverse z-transform (left-sided, right-sided, or two-sided sequences) associated with each of the four ROCs. It is not necessary to compute the exact inverse transform.

6.9 Let the z-transform of a sequence $x[n]$ be $X(z)$, with $\mathcal{R}_x$ denoting its ROC. Express the z-transforms of the real and imaginary parts of $x[n]$ in terms of $X(z)$. Show also their respective ROCs.

6.10 The z-transform $X(z)$ of the length-9 sequence of Problem 3.38 is sampled at seven points $\omega_k = 2\pi k/7$, $0 \le k \le 6$, on the unit circle yielding the frequency samples

$$\tilde{X}[k] = X(z)|_{z=e^{j2\pi k/7}}, \qquad 0 \le k \le 6.$$

Determine, without evaluating $\tilde{X}[k]$, the periodic sequence $\tilde{x}[n]$ whose discrete Fourier series coefficients are given by $\tilde{X}[k]$. What is the period of $\tilde{x}[n]$?

6.11 Repeat Problem 6.10 for the length-9 sequence of Problem 3.39.

6.12 Let $X(z)$ denote the z-transform of the length-12 sequence $x[n]$ of Problem 5.34. Let $X_0[k]$ represent the samples of $X(z)$ evaluated on the unit circle at nine equally spaced points given by $z = e^{j(2\pi k/9)}$, $0 \le k \le 8$, i.e.,

$$X_0[k] = X(z)|_{z=e^{j(2\pi k/9)}}, \qquad 0 \le k \le 8.$$

Determine the 9-point IDFT $x_0[n]$ of $X_0[k]$ without computing the latter function.

6.13 Consider the causal sequence $x[n] = (-0.5)^n\mu[n]$, with a z-transform given by $X(z)$.
(a) Determine the inverse z-transform of $X(z^3)$ without computing $X(z)$.
(b) Determine the inverse z-transform of $(1 + z^{-2})X(z^3)$ without computing $X(z)$.

6.14 Determine the z-transforms of the sequences of Problem 3.18 and their ROCs. Show that the ROC includes the unit circle for each z-transform. Evaluate the z-transform evaluated on the unit circle for each sequence and show that it is precisely the DTFT of the respective sequence computed in Problem 3.18.

6.15 Determine the z-transforms of the sequences of Problem 3.19 and their ROCs. Show that the ROC includes the unit circle for each z-transform. Evaluate the z-transform evaluated on the unit circle for each sequence and show that it is precisely the DTFT of the respective sequence computed in Problem 3.19.

6.16 Evaluate the linear convolutions of Problem 2.50 using the polynomial multiplication method.

6.17 Prove Eq. (6.73).

6.18 Evaluate the linear and circular convolutions of the sequences $g[n]$ and $h[n]$ of Problem 5.45 using the polynomial multiplication method. Verify your results in MATLAB using the functions `conv` and `circonv`.


circonv.m

6.19 Consider a rational z-transform $G(z) = P(z)/D(z)$, where $P(z)$ and $D(z)$ are polynomials in z^{-1}. Let ρ_ℓ denote the residue of $G(z)$ at a simple pole at $z = \lambda_\ell$. Show that

$$\rho_\ell = -\lambda_\ell \frac{P(z)}{D'(z)}\bigg|_{z=\lambda_\ell},$$

where $D'(z) = \frac{dD(z)}{dz^{-1}}$.

6.20 Each one of following z-transforms

$$X_a(z) = \frac{3z}{z^2 + 0.3z - 0.18}, \qquad X_b(z) = \frac{3z^2 + 0.1z + 0.87}{(z + 0.6)(z - 0.3)^2}$$

has three ROCs. Evaluate their respective inverse z-transforms corresponding to each ROC.

6.21 Consider the z-transform $G(z)$ of Eq. (6.13), with $M < N$. If $G(z)$ has only simple poles, show that p_0/d_0 is equal to the sum of the residues in the partial-fraction expansion of $G(z)$ [Mit98].

6.22 Show that the inverse z-transform $h[n]$ of the following rational z-transform

$$H(z) = \frac{1}{1 - 2r(\cos\theta)z^{-1} + r^2 z^{-2}}, \qquad |z| > r > 0$$

is given by

$$h[n] = \frac{r^n \sin(n+1)\theta}{\sin\theta} \cdot \mu[n].$$

6.23 Determine z-transform of each of the following left-sided sequences:
(a) $x[n] = \alpha^n \mu[-n-1]$, (b) $y[n] = (n+1)\alpha^n \mu[-n-1]$.

6.24 Determine the inverse z-transforms, $x_1[n]$ and $x_2[n]$, of the following rational z-transforms
(a) $X_1(z) = \frac{1}{1 - z^{-3}}$, $|z| > 1$, (b) $X_2(z) = \frac{1}{1 - z^{-4}}$, $|z| > 1$,
by expanding each in a power series and computing the inverse z-transform of the individual terms in the power series. Compare the results with that obtained using a partial-fraction approach.

6.25 Determine the inverse z-transforms of the following z-transforms:
(a) $X_1(z) = \log(1 - \alpha z^{-1})$, $|z| > |\alpha|$, (b) $X_2(z) = \log\left(\frac{\alpha - z^{-1}}{\alpha}\right)$, $|z| > 1/|\alpha|$,
(c) $X_3(z) = \log\left(\frac{1}{1 - \alpha z^{-1}}\right)$, $|z| > |\alpha|$, (d) $X_4(z) = \log\left(\frac{\alpha}{\alpha - z^{-1}}\right)$, $|z| > 1/|\alpha|$.

6.26 The z-transform of a right-sided sequence $h[n]$ is given by

$$H(z) = \frac{z + 1.7}{(z - 0.3)(z + 0.5)}.$$

Find its inverse z-transform $h[n]$ via the partial-fraction approach. Verify the partial fraction expansion using MATLAB.

6.27 Prove the following properties of the z-transform listed in Table 6.5: (a) conjugation, (b) time-reversal, (c) linearity, (d) time-shifting, (e) multiplication by an exponential sequence, and (f) differentiation.

6.28 A generalization of the DFT concept leads to the *nonuniform discrete Fourier transform* (NDFT) $X_{\text{NDFT}}[k]$, defined by [Bag98]

$$X_{\text{NDFT}}[k] = X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n}, \qquad 0 \le k \le N-1, \tag{6.109}$$

where $z_0, z_1, \ldots, z_{N-1}$, are N distinct points located arbitrarily in the z-plane. The NDFT has been applied to the efficient design of digital filters, antenna array design, and dual-tone multifrequency detection [Bag98]. The NDFT can be expressed in a matrix form as

$$\begin{bmatrix} X_{\text{NDFT}}[0] \\ X_{\text{NDFT}}[1] \\ \vdots \\ X_{\text{NDFT}}[N-1] \end{bmatrix} = \mathbf{D}_N \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \tag{6.110}$$

where

$$\mathbf{D}_N = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & \cdots & z_0^{-(N-1)} \\ 1 & z_1^{-1} & z_1^{-2} & \cdots & z_1^{-(N-1)} \\ 1 & z_2^{-1} & z_2^{-2} & \cdots & z_2^{-(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_{N-1}^{-1} & z_{N-1}^{-2} & \cdots & z_{N-1}^{-(N-1)} \end{bmatrix} \tag{6.111}$$

is the $N \times N$ NDFT matrix. The matrix $\mathbf{D}_N$ is known as the *Vandermonde matrix*. Show that it is nonsingular provided the N sampling points z_k are distinct. In which case, the inverse NDFT is given by

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \mathbf{D}_N^{-1} \begin{bmatrix} X_{\text{NDFT}}[0] \\ X_{\text{NDFT}}[1] \\ \vdots \\ X_{\text{NDFT}}[N-1] \end{bmatrix}. \tag{6.112}$$

6.29 In general, for large N, the Vandermonde matrix is usually ill-conditioned (except for the case when the NDFT reduces to the conventional DFT), and a direct inverse computation is not advisable. A more efficient way is to directly determine the z-transform $X(z)$,

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}, \tag{6.113}$$

and hence, the sequence $x[n]$, from the given N-point NDFT $X_{\text{NDFT}}[k]$ by using some type of polynomial interpolation method [Bag98]. One popular method is the Lagrange interpolation formula, which expresses $X(z)$ as

$$X(z) = \sum_{k=0}^{N-1} \frac{I_k(z)}{I_k(z_k)} X_{\text{NDFT}}[k], \tag{6.114}$$

where

$$I_k(z) = \prod_{\substack{i=0 \\ i \neq k}}^{N-1} (1 - z_i z^{-1}). \tag{6.115}$$

Consider the z-transform $X(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}$ of a length-4 sequence $x[n]$. By evaluating $X(z)$ at $z_0 = -1/2$, $z_1 = 1$, $z_2 = 1/2$, and $z_3 = 1/3$, determine the 4-point NDFT of $x[n]$ and then use the Lagrange interpolation method to show that $X(z)$ can be uniquely determined from these NDFT samples.

6.30 Consider a sequence $x[n]$ with a z-transform $X(z)$. Define a new z-transform $\hat{X}(z)$ given by the complex natural logarithm of $X(z)$; that is, $\hat{X}(z) = \ln X(z)$.The inverse z-transform of $\hat{X}(z)$ to be denoted by $\hat{x}[n]$ is called the *complex cepstrum* of $x[n]$ [Tri79]. Assume that the ROCs of both $X(z)$ and $\hat{X}(z)$ include the unit circle.
(a) Relate the DTFT $X(e^{j\omega})$ of $x[n]$ to the DTFT $\hat{X}(e^{j\omega})$ of its complex cepstrum $\hat{x}[n]$.
(b) Show that the complex cepstrum of a real sequence is a real-valued sequence.
(c) Let $\hat{x}_{ev}[n]$ and $\hat{x}_{od}[n]$ denote, respectively, the even and odd parts of a real-valued complex cepstrum $\hat{x}[n]$. Express $\hat{x}_{ev}[n]$ and $\hat{x}_{od}[n]$ in terms of $X(e^{j\omega})$, the DTFT of $x[n]$.

6.31 Determine the complex cepstrum $\hat{x}[n]$ of a sequence $x[n] = a\delta[n] + b\delta[n-1]$, where $|b/a| < 1$. Comment on your results.

6.32 Let $x[n]$ be a sequence with a rational z-transform $X(z)$ given by

$$X(z) = K \frac{\prod_{k=1}^{N_\alpha}(1 - \alpha_k z^{-1}) \prod_{k=1}^{N_\gamma}(1 - \gamma_k z)}{\prod_{k=1}^{N_\beta}(1 - \beta_k z^{-1}) \prod_{k=1}^{N_\delta}(1 - \delta_k z)},$$

where α_k and β_k are the zeros and poles of $X(z)$ that are strictly inside the unit circle and $1/\gamma_k$ and $1/\delta_k$ are the zeros and poles of $X(z)$ that are strictly outside the unit circle [Rab78].
(a) Determine the exact expression for the complex cepstrum $\hat{x}[n]$ of $x[n]$.
(b) Show that $\hat{x}[n]$ is a decaying bounded sequence as $|n| \to \infty$.
(c) If $\alpha_k = \beta_k = 0$, show that $\hat{x}[n]$ is an anticausal sequence.
(d) If $\gamma_k = \delta_k = 0$, show that $\hat{x}[n]$ is a causal sequence.

6.33 Let $x[n]$ be a sequence with a rational z-transform $X(z)$ with poles and zeros strictly inside the unit circle. Show that the complex cepstrum $\hat{x}[n]$ of $x[n]$ can be computed using the recursion relation [Rab78]:

$$\hat{x}[n] = \begin{cases} 0, & n < 0, \\ \log(x[0]), & n = 0, \\ \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \frac{k}{n} \cdot \hat{x}[k] \cdot \frac{x[n-k]}{x[0]}, & n > 0. \end{cases}$$

6.34 The magnitude response of a digital filter with a real-coefficient transfer function $H(z)$ is shown in Figure P6.1. Plot the magnitude response of the filter $H(z^5)$.

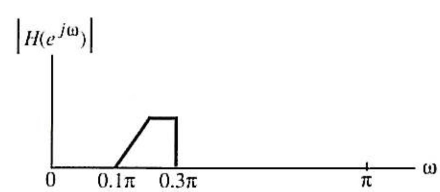

Figure P6.1

6.35 In this problem, we consider the determination of a real rational, causal, stable discrete-time transfer function

$$H(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{N} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

from the specified real part of its frequency response [Dut83]:

$$H_{re}(e^{j\omega}) = \frac{\sum_{i=0}^{N} a_i \cos(i\omega)}{\sum_{i=0}^{N} b_i \cos(i\omega)} = \frac{A(e^{j\omega})}{B(e^{j\omega})}. \tag{6.116}$$

(a) Show that

$$H_{re}(e^{j\omega}) = \tfrac{1}{2}\left[H(z) + H(z^{-1})\right]\Big|_{z=e^{j\omega}} = \tfrac{1}{2}\left[\frac{P(z)D(z^{-1})+P(z^{-1})D(z)}{D(z)D(z^{-1})}\right]\Big|_{z=e^{j\omega}}. \tag{6.117}$$

(b) Comparing Eqs. (6.116) and (6.117), we get

$$\begin{aligned} B(e^{j\omega}) &= D(z)D(z^{-1})\Big|_{z=e^{j\omega}} \\ A(e^{j\omega}) &= \tfrac{1}{2}\left[P(z)D(z^{1}) + P(z^{-1})D(z)\right]\Big|_{z=e^{j\omega}}. \end{aligned} \tag{6.118}$$

The spectral factor $D(z)$ can be determined, except for the scale factor K, from the roots of $B(z) = B(e^{j\omega})\big|_{z=e^{j\omega}}$ inside the unit circle. Show that

$$K = \sqrt{B(1)} \Big/ \prod_{i=1}^{N}(1 - z_i).$$

(c) To determine $P(z)$, Eq. (6.118) can be rewritten through analytic continuation as

$$A(z) = \tfrac{1}{2}\left[P(z)D(z^{-1}) + P(z^{-1})D(z)\right].$$

Substituting the polynomial forms of $P(z)$ and $D(z)$ and equating coefficients of $(z^i + z^{-i})/2$ on both sides of the above equation, we arrive at a set of $N+1$ equations that can be solved for the numerator coefficients $\{p_i\}$. Using the above approach, determine $H(z)$ for which

$$H_{re}(e^{j\omega}) = \frac{1 + \cos\omega + \cos 2\omega}{17 - 8\cos 2\omega}.$$

6.36 Let $H(z)$ be the transfer function of a causal stable LTI discrete-time system. Let $G(z)$ be the transfer function obtained by replacing z^{-1} in $H(z)$ with a stable allpass function $A(z)$; that is, $G(z) = H(z)|_{z^{-1}=A(z)}$. Show that $G(1) = H(1)$ and $G(-1) = H(-1)$.

6.37 Consider the digital filter structure of Figure P6.2, where

$$H_1(z) = 1.2 + 3.3z^{-1} + 0.7z^{-2}, \quad H_2(z) = -4.1 - 2.5z^{-1} + 0.9z^{-2}, \quad H_3(z) = 2.3 + 4.3z^{-1} + 0.8z^{-2}.$$

Determine the transfer function $H(z)$ of the composite filter.

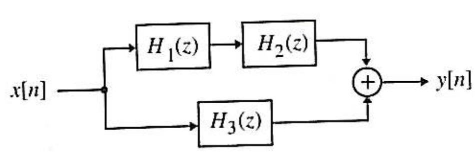

Figure P6.2

6.38 Determine the transfer function of each of the following causal LTI discrete-time systems described by the difference equations. Express each transfer function in a factored form and sketch its pole-zero plot. Is the corresponding system BIBO stable?

(a) $y[n] = 5x[n] + 9.5x[n-1] + 1.4x[n-2] - 24x[n-3] + 0.1y[n-1] - 0.14y[n-2] - 0.49y[n-3]$.

(b) $y[n] = 5x[n] + 16.5x[n-1] + 14.7x[n-2] - 22.04x[n-3] - 33.6x[n-4] + 0.5y[n-1] - 0.1y[n-2]$ $-0.3y[n-3] + 0.0936y[n-4]$.

6.39 Determine the expression for the impulse response $\{h[n]\}$ of the following causal IIR transfer function:

$$H(z) = \frac{4.5 - 1.3z^{-1} + 1.12z^{-2}}{(1 + 0.5z^{-1} + 0.3z^{-2})(1 - 0.4z^{-1})}.$$

6.40 The transfer function of a causal LTI discrete-time system is given by

$$H(z) = \frac{1 - 3.3z^{-1} + 0.36z^{-2}}{1 + 0.3z^{-1} - 0.18z^{-2}}.$$

(a) Determine the impulse response $h[n]$ of the above system.

(b) Determine the output $y[n]$ of the above system for all values of n for an input

$$x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n].$$

6.41 Using z-transform methods, determine the explicit expression for the output $y[n]$ of each of the following causal LTI discrete-time systems, with impulse responses and inputs as indicated:

(a) $h[n] = (-0.4)^n \mu[n]$, $x[n] = (0.2)^n \mu[n]$, (b) $h[n] = (-0.2)^n \mu[n]$, $x[n] = (-0.2)^n \mu[n]$.

6.42 Using z-transform methods, determine the explicit expression for the impulse response $h[n]$ of a causal LTI discrete-time system that develops an output $y[n] = 2(-0.3)^n \mu[n]$ for an input $x[n] = 4(0.6)^n \mu[n]$.

6.43 A causal LTI discrete-time system is described by the difference equation

$$y[n] = 0.2y[n-1] + 0.08y[n-2] + 2x[n],$$

where $x[n]$ and $y[n]$ are, respectively, the input and the output sequences of the system.

(a) Determine the transfer function $H(z)$ of the system.

(b) Determine the impulse response $h[n]$ of the system.

(c) Determine the step response $s[n]$ of the system.

6.44 Determine the frequency response $H(e^{j\omega})$ of the transfer function

$$H(z) = \frac{1 - z^{-2}}{1 - (1 + \alpha)\cos(\omega_c)z^{-1} + \alpha z^{-2}}.$$

Show that the magnitude response $|H(e^{j\omega})|$ assumes its maximum value of $2/(1-\alpha)$ at $\omega = \omega_c$.

6.45 Determine a closed-form expression for the frequency response $H(e^{j\omega})$ of the LTI discrete-time system characterized by an impulse response

$$h[n] = \delta[n] - \alpha\delta[n - R], \qquad (6.119)$$

where $|\alpha| < 1$. What are the maximum and the minimum values of its magnitude response? How many peaks and dips of the magnitude response occur in the range $0 \le \omega < 2\pi$? What are the locations of the peaks and the dips? Sketch the magnitude and the phase responses for $R = 6$.

6.46 Determine a closed-form expression for the frequency response $G(e^{j\omega})$ of the LTI discrete-time system characterized by an impulse response

$$g[n] = h[n] \circledast h[n] \circledast h[n], \tag{6.120}$$

where $h[n]$ is given by Eq. (6.119).

6.47 Determine a closed-form expression for the frequency response $G(e^{j\omega})$ of an LTI discrete-time system with an impulse response given by

$$g[n] = \begin{cases} \alpha^n, & 0 \le n \le M-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\alpha| < 1$. What is the relation of $G(e^{j\omega})$ to $H(e^{j\omega})$ of Eq. (3.98)? Scale the impulse response by multiplying it with a suitable constant so that the dc value of the magnitude response is unity.

6.48 Determine the expression for the frequency response $H(e^{j\omega})$ of a causal IIR LTI discrete-time system characterized by the input–output relation

$$y[n] = x[n] + \alpha y[n-R], \qquad |\alpha| < 1,$$

where $y[n]$ and $x[n]$ denote, respectively, the output and the input sequences. Determine the maximum and the minimum values of its magnitude response. How many peaks and dips of the magnitude response occur in the range $0 \le \omega < 2\pi$? What are the locations of the peaks and the dips? Sketch the magnitude and the phase responses for $R = 6$.

6.49 An IIR LTI discrete-time system is described by the difference equation

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2],$$

where $y[n]$ and $x[n]$ denote, respectively, the output and the input sequences. Determine the expression for its frequency response. For what values of the constants b_i will the magnitude response be a constant for all values of ω?

6.50 Determine the input–output relation of a factor-of-2 up-sampler in the frequency domain. The time-domain input–output relation of the factor-of-L up-sampler is given by Eq. (2.20).

6.51 Consider an LTI discrete-time system with an impulse response $h[n] = (0.5)^n \mu[n]$. Determine the frequency response $H(e^{j\omega})$ of the system and evaluate its value at $\omega = \pm\pi/5$. What is the steady-state output $y[n]$ of the system for an input $x[n] = \cos(\pi n/5)\mu[n]$?

6.52 Let $H(z)$ be the transfer function of a causal, stable LTI discrete-time system. Consider the transfer function $G(z) = H(z)|_{z=F(z)}$. What are the conditions that need to be satisfied by the transformation $F(z)$ so that $G(z)$ remains stable?

6.53 Determine the z-transform $F(z)$ of the Fibonacci sequence $\{f[n]\}$ of Problem 2.70. Evaluate the inverse z-transform of $F(z)$.

6.54 The *time constant* K of an LTI stable causal discrete-time system with an impulse response $h[n]$ is given by the value of the total time interval n at which the partial energy of the impulse response is within 95% of the total energy; that is,

$$\sum_{n=0}^{K} |h[n]|^2 = 0.95 \sum_{n=0}^{\infty} |h[n]|^2 .$$

Determine the time constant K of the first-order causal transfer function $H(z) = 1/(1 - \beta z^{-1})$, $|\beta| < 1$.

6.55 Figures P6.3(a) and P6.3(b) show, respectively, the DPCM (*differential pulse-code modulation*) coder and decoder often employed for the compression of digital signals [Jay84]. The linear predictor $P(z)$ in the encoder develops a prediction $\hat{x}[n]$ of the input signal $x[n]$, and the difference signal $d[n] = x[n] - \hat{x}[n]$ is quantized by the quantizer Q developing the quantized output $u[n]$, which is represented with fewer bits than that of $x[n]$. The output of the encoder is transmitted over a channel to the decoder. In the absence of any errors due to transmission and quantization, the input $v[n]$ to the decoder is equal to $u[n]$, and the decoder generates the output $y[n]$, which is equal to the input $x[n]$. Determine the transfer function $H(z) = U(z)/X(z)$ of the encoder in the absence of any quantization and the transfer function $G(z) = Y(z)/V(z)$ of the decoder for the case of each of the following predictors, and show that $G(z)$ is the inverse of $H(z)$ in each case.

(a) $P(z) = h_1 z^{-1}$, and (b) $P(z) = h_1 z^{-1} + h_2 z^{-2}$.

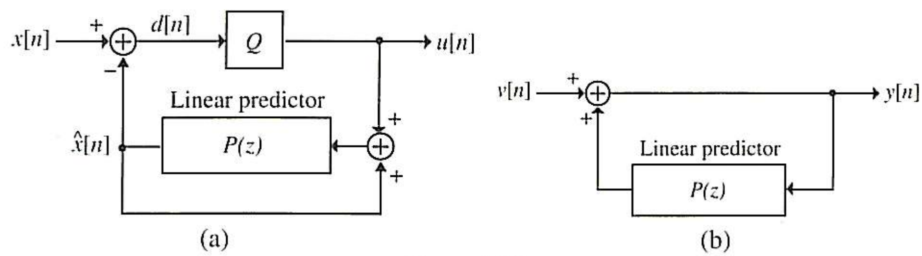

Figure P6.3

6.56 Consider the discrete-time system of Figure P6.4. For $H_0(z) = 1 + \alpha z^{-1}$, find a suitable $F_0(z)$ so that the output $y[n]$ is a delayed and scaled replica of the input.

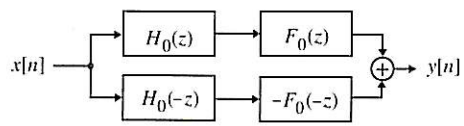

Figure P6.4

6.57 A causal stable LTI discrete-time system is characterized by an impulse response $h_1[n] = 1.2\delta[n] + 0.5(-0.5)^n \mu[n] + -0.6(0.2)^n \mu[n]$. Determine the impulse response $h_2[n]$ of its inverse system, which is causal and stable.

6.58 Show that the group delay $\tau_g(\omega)$ of an LTI transfer function $H(z)$ can be expressed as [Fot2001]

$$\tau_g(\omega) = \left. \frac{T(z) + T(z^{-1})}{2} \right|_{z=e^{j\omega}}, \tag{6.121}$$

where $T(z) = z \frac{dH(z)/dz}{H(z)}$.

6.10 MATLAB Exercises

Program 6_1.m

M 6.1 Using Program 6_1, determine the factored form of the following z-transforms:

(a) $G_1(z) = \dfrac{2z^4 - 5z^3 + 13.48z^2 - 7.78z + 9}{4z^4 + 7.2z^3 + 20z^2 - 0.8z + 8}$,

(b) $G_2(z) = \dfrac{5z^4 + 3.5z^3 + 21.5z^2 - 4.6z + 18}{5z^4 + 15.5z^3 + 31.7z^2 + 22.52z + 4.8}$, and show their pole-zero plots. Determine all possible ROCs of each of the above z-transforms, and describe the type of their inverse z-transforms (left-sided, right-sided, two-sided sequences) associated with each of the ROCs.

Program 6_3.m

M 6.2 Using Program 6_3, determine the partial-fraction expansions of the z-transforms listed in Problem 6.20, and then determine their inverse z-transforms.

Program 6_4.m

M 6.3 Using Program 6_4, determine the z-transform as a ratio of two polynomials in z^{-1} from each of the partial-fraction expansions listed below:

(a) $X_1(z) = 3 - \dfrac{4}{5+z^{-1}} - \dfrac{7}{6-z^{-1}}, \quad |z| > 0.2,$

(b) $X_2(z) = -2.5 + \dfrac{3}{1+0.4z^{-1}} - \dfrac{1.4+z^{-1}}{1+0.6z^{-2}}, \quad |z| > 0.25,$

(c) $X_3(z) = \dfrac{-4}{(4+2z^{-1})^2} + \dfrac{6}{4+2z^{-1}} + \dfrac{5}{1+0.64z^{-2}}, \quad |z| > 0.8,$

(d) $X_4(z) = -5 + \dfrac{2}{4+3z^{-1}} + \dfrac{z^{-1}}{4+3z^{-1}+0.9z^{-2}}, \quad |z| > 0.75.$

Program 6_5

M 6.4 Using Program 6_5, determine the first 30 samples of the inverse z-transforms of the rational z-transforms determined in Problem M6.3. Show that these samples are identical to those obtained by explicitly evaluating the exact inverse z-transforms.

M 6.5 Repeat Problem 6.57 using MATLAB.

M 6.6 Write a MATLAB program to compute the NDFT and the inverse NDFT using the Lagrange interpolation method. Verify your program by computing the NDFT of a length-20 sequence and reconstructing the sequence from its computed NDFT.