

Chapter (3)

Continuous time Fourier Transform

Discrete time Fourier Transform

Four Fourier Representations

Time Domain	Periodic	Non-periodic	
Discrete	<p>Discrete-Time Fourier Series</p> $\tilde{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N}$ $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$	<p>Discrete-Time Fourier Transform</p> $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$	Periodic
Continuous	<p>Fourier Series</p> $X[k] = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-j2\pi kt/T} dt$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T}$	<p>Fourier Transform</p> $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$	Non-periodic
	Discrete	Continuous	Freq. Domain

3.2 The Discrete-Time Fourier Transform

The *discrete-time Fourier transform* (DTFT) of a discrete-time sequence $x[n]$ is a representation of the sequence in terms of the complex exponential sequence $\{e^{j\omega n}\}$, where ω is the real frequency variable. If there is no ambiguity, for brevity often the discrete-time Fourier transform is called simply the *Fourier transform* (FT). The Fourier transform representation of a sequence, if it exists, is unique, and the original sequence can be computed from its transform representation by an inverse transform operation. We first define the forward transform and derive its inverse transform. We then describe the condition for its existence and review its important properties.

The *discrete-time Fourier transform* $X(e^{j\omega})$ of a sequence $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

$\mathcal{F}\{x[n]\}$

the *analysis equation*,

$\mathcal{F}^{-1}\{X(e^{j\omega})\}$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

synthesis equation,

(3.16)

called the *inverse discrete-time Fourier transform*. It should be noted that even though the integration in Eq. (3.16) can be carried out over any interval of duration 2π , it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.16) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi} e^{j\omega n} d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π .

 $\mathcal{F}\{x[n]\}$

to denote the $X(e^{j\omega})$ of the sequence $x[n]$. Likewise, we shall use the operator symbol

 $\mathcal{F}^{-1}\{X(e^{j\omega})\}$

to denote the inverse Fourier transform $x[n]$ of the transform $X(e^{j\omega})$. A discrete-time Fourier transform pair will be denoted as

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}).$$

(3.17)

Example ①

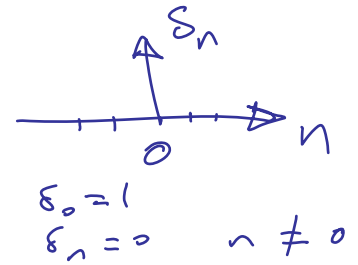
$$x_n = \delta_n$$

$$X(e^{j\omega}) = ??$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_n e^{j\omega n}$$

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta_n e^{j\omega n} = 1$$

$$\therefore \delta_n \xrightarrow{\text{DTFT}} 1$$



Example ②

$$x_n = \alpha^n u(n) \quad |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

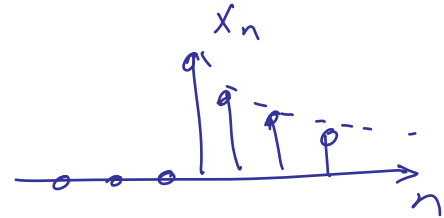


Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
1, $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\alpha^n \mu[n]$, $(\alpha < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$
$(n + 1)\alpha^n \mu[n]$, $(\alpha < 1)$	$\frac{1}{(1 - \alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$, $(-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega \leq \omega_c, \\ 0, & \omega_c < \omega \leq \pi \end{cases}$

3.3 Discrete-Time Fourier Transform Theorems

There are a number of important theorems of the discrete-time Fourier transform that are useful in digital signal processing applications. These theorems can be used to determine the Fourier transforms of sequences obtained by combining sequences with known transforms. We review these theorems in this section. For compactness, the theorems are stated using the operator notation introduced in Eq. (3.17), and we shall make use of the following Fourier transform pairs:

$$g[n] \xleftrightarrow{\mathcal{F}} G(e^{j\omega}), \quad (3.56a)$$

$$h[n] \xleftrightarrow{\mathcal{F}} H(e^{j\omega}). \quad (3.56b)$$

The proofs of most of the theorems given here are quite straightforward and are left as exercises.

Convolution Theorem

The Fourier transform $Y(e^{j\omega})$ of the convolution sum of two sequences, $y[n] = g[n] \otimes h[n]$, is given by the product of their Fourier transforms $G(e^{j\omega})H(e^{j\omega})$; that is,

$$g[n] \otimes h[n] \xleftrightarrow{\mathcal{F}} G(e^{j\omega})H(e^{j\omega}). \quad (3.65)$$

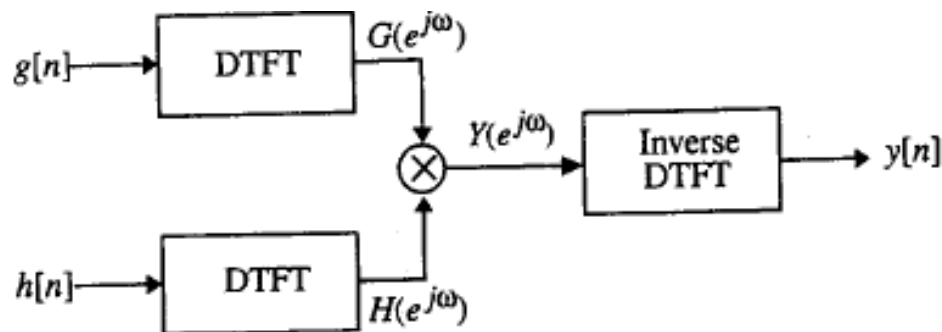


Figure 3.9: Linear convolution using the discrete-time Fourier transform.

Modulation Theorem

The Fourier transform $Y(e^{j\omega})$ of the product of two sequences given by $y[n] = g[n]h[n]$ is given by the convolution integral of their Fourier transforms $\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)}) d\theta$; that is,

$$g[n]h[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)}) d\theta. \quad (3.66)$$

The modulation theorem plays a key role in the amplitude modulation scheme used in digital communications. The modulation theorem is also known as the *windowing theorem*. One application of this theorem, to be considered later in Section 10.2, is in the design of linear-phase FIR filter based on the windowing of the impulse response of an ideal linear-phase filter with a doubly-infinite impulse response.

Linearity Theorem

Consider a sequence $x[n] = \alpha g[n] + \beta h[n]$ obtained by a linear combination of $g[n]$ and $h[n]$, where α and β are arbitrary constants. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is then given by $\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$; that is,

$$\alpha g[n] + \beta h[n] \xleftrightarrow{\mathcal{F}} \alpha G(e^{j\omega}) + \beta H(e^{j\omega}). \quad (3.57)$$

It should be noted that the Fourier transforms of the conjugate-symmetric and conjugate-antisymmetric parts of a sequence derived in Eqs. (3.34) and (3.35) made use of the linearity theorem.

Time-Reversal Theorem

The Fourier transform of the time-reversed sequence $g[-n]$ is given by $G(e^{-j\omega})$; that is,

$$g[-n] \xleftrightarrow{\mathcal{F}} G(e^{-j\omega}). \quad (3.58)$$

Time-Shifting Theorem

The Fourier transform of the delayed sequence $x[n] = g[n - n_0]$, with n_0 an integer, is given by $X(e^{j\omega}) = e^{-j\omega n_0} G(e^{j\omega})$; that is,

$$g[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} G(e^{j\omega}). \quad (3.59)$$

It follows from Equation 3.59 that since $|e^{-j\omega n_0}| = 1$, $|G(e^{j\omega})| = |X(e^{j\omega})|$; that is, the magnitude spectrum is unchanged by shifting a signal in time.

Frequency-Shifting Theorem

The Fourier transform of a sequence $x[n] = e^{j\omega_0 n} g[n]$ is given by $X(e^{j\omega}) = G(e^{j(\omega - \omega_0)})$; that is,

$$e^{j\omega_0 n} g[n] \xleftrightarrow{\mathcal{F}} G(e^{j(\omega - \omega_0)}). \quad (3.63)$$

Differentiation-in-Frequency Theorem

The Fourier transform of a sequence $x[n] = n g[n]$ is given by $X(e^{j\omega}) = j \frac{dG(e^{j\omega})}{d\omega}$; that is,

$$n g[n] \xleftrightarrow{\mathcal{F}} j \frac{dG(e^{j\omega})}{d\omega}. \quad (3.64)$$

Table 3.4: Discrete-time Fourier transform theorems.

Theorem	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-reversal	$g[-n]$	$G(e^{-j\omega})$
Time-shifting	$g[n - n_0]$	$e^{-j\omega n_0} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_0 n} g[n]$	$G(e^{j(\omega - \omega_0)})$
Differentiation- in-frequency	$ng[n]$	$j \frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \otimes h[n]$	$G(e^{j\omega}) H(e^{j\omega})$
Modulation	$g[n] h[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega - \theta)}) d\theta$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} g[n] h^*[n]$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) H^*(e^{j\omega}) d\omega$

3.6 DTFT Computation Using MATLAB

The *Signal Processing Toolbox* in MATLAB includes a number of M-files to aid in the DTFT-based analysis of discrete-time signals. Specifically, the functions that can be used are `freqz`, `abs`, `angle`, and `unwrap`. In addition, the built-in MATLAB functions `real` and `imag` are also useful in some applications.

FREQZ Digital filter frequency response.

[H,W] = FREQZ(B,A,N) returns the N-point complex frequency response vector H and the N-point frequency vector W in radians/sample of the filter:

$$H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})} = \frac{b(1) + b(2)e^{-j\omega} + \dots + b(m+1)e^{-j\omega m}}{a(1) + a(2)e^{-j\omega} + \dots + a(n+1)e^{-j\omega n}}$$

given numerator and denominator coefficients in vectors B and A.

The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If N isn't specified, it defaults to 512.

In this book, many of the Fourier transforms we shall encounter are rational functions in $e^{-j\omega}$, that is, ratios of polynomials in $e^{-j\omega}$, and are of the form

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{D(e^{j\omega})} = \frac{p_0 + p_1 e^{-j\omega} + \dots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \dots + d_N e^{-j\omega N}}. \quad (3.55)$$

DTFT Computation Using MATLAB

- For example, the statement

```
H = freqz(num, den, w)
```

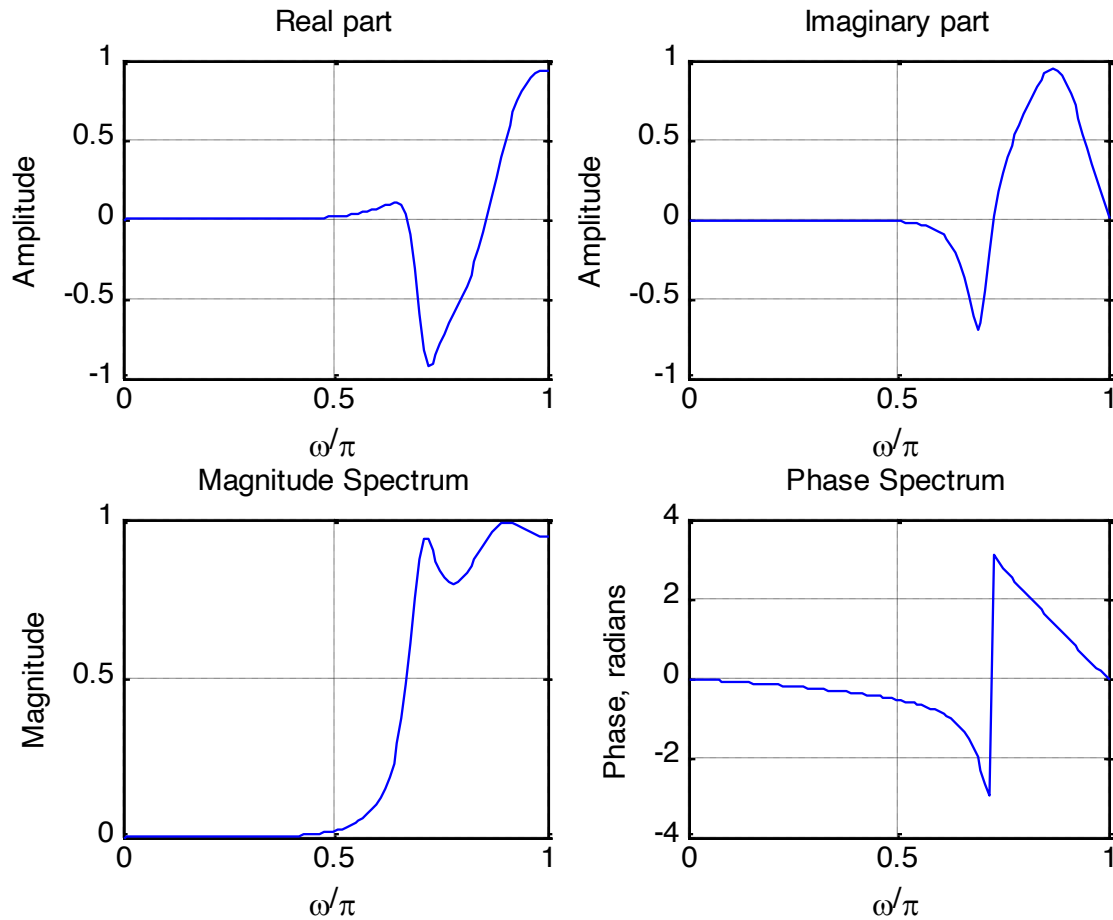
returns the frequency response values as a vector H of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector `w`

```

% Program 3_1
% Discrete-Time Fourier Transform Computation
%
% Read in the desired number of frequency samples
k = input('Number of frequency points = ');
% Read in the numerator and denominator coefficients
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
% Compute the frequency response
w = 0:pi/(k-1):pi;
h = freqz(num, den, w);

% Plot the frequency response
subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')

```



Number of frequency points = 100

Numerator coefficients = [0.008 -0.033 0.05 -0.033 0.008]

Denominator coefficients = [1 2.37 2.7 1.6 0.41]

3.8 The Frequency Response of an LTI Discrete-Time System

Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies. Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property of the system. Since a sinusoidal signal can be expressed in terms of an exponential signal, the response of the LTI system to an exponential input is of practical interest. This leads to the concept of frequency response, a transform-domain representation of the LTI discrete-time system. We first define the frequency response, investigate its properties, and describe some of its applications. The computation of the time-domain representation of the LTI system from its frequency response is outlined.

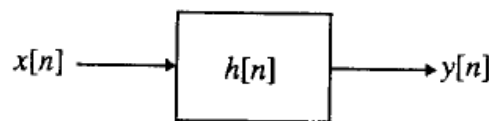


Figure 3.12: An LTI discrete-time system.

3.8.1 Definition

An important property of an LTI system is that for certain types of input signals, called *eigenfunctions*, the output signal is the input signal multiplied by a complex constant. We consider here one such eigenfunction as the input. Recall from Section 2.5.1, the input-output relationship of an LTI discrete-time system as shown in Figure 3.12, with an impulse response $h[n]$, is given by the convolution sum of Eq. (2.73b) and is of the form

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \quad (3.81)$$

where $y[n]$ and $x[n]$ are, respectively, the output and the input sequences. Now, if the input $x[n]$ is a complex exponential sequence of the form

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty, \quad (3.82)$$

then, from Eq. (3.81), the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}, \quad (3.83)$$

which can be rewritten as

$$y[n] = H(e^{j\omega})e^{j\omega n}, \quad (3.84)$$

where we have used the notation

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}. \quad (3.85)$$

It can be seen from Eq. (3.85) that for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$. Thus, $e^{j\omega n}$ is an eigenfunction of the system.

The quantity $H(e^{j\omega})$ defined above is called the *frequency response* of the LTI discrete-time system, and it provides a frequency-domain description of the system. Note from Eq. (3.85) that $H(e^{j\omega})$ is precisely the Fourier transform of the impulse response $h[n]$ of the system.

Just like any other discrete-time Fourier transform, in general, $H(e^{j\omega})$ is also a complex function of ω with a period 2π and can be expressed in terms of its real and imaginary parts or its magnitude and phase. Thus,

$$\begin{aligned} H(e^{j\omega}) &= H_{\text{re}}(e^{j\omega}) + jH_{\text{im}}(e^{j\omega}) \\ &= |H(e^{j\omega})|e^{j\theta(\omega)}, \end{aligned}$$

where $H_{\text{re}}(e^{j\omega})$ and $H_{\text{im}}(e^{j\omega})$ are, respectively, the real and imaginary parts of $H(e^{j\omega})$, and

$$\theta(\omega) := \arg\{H(e^{j\omega})\}. \quad (3.86)$$

The quantity $|H(e^{j\omega})|$ is called the *magnitude response* and the quantity $\theta(\omega)$ is called the *phase response* of the LTI discrete-time system. Design specifications for the discrete-time systems, in many applications, are given in terms of the magnitude response or the phase response or both. In some cases, the magnitude function is specified in decibels as defined below:

$$\mathcal{G}(\omega) = 20 \log_{10} |H(e^{j\omega})| \quad \text{dB}, \quad (3.87)$$

where $\mathcal{G}(\omega)$ is called the *gain function*. The negative of the gain function, $\mathcal{A}(\omega) = -\mathcal{G}(\omega)$, is called the *attenuation* or *loss function*.

3.8.2 Frequency-Domain Characterization of the LTI Discrete-Time System

We now derive the frequency-domain representation of an LTI discrete-time system. If $Y(e^{j\omega})$ and $X(e^{j\omega})$ denote the Fourier transforms of the output and input sequences, $y[n]$ and $x[n]$, respectively, then applying the convolution theorem of Table 3.4 to Eq. (3.81), we arrive at

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \quad (3.88)$$

where $H(e^{j\omega})$ is the frequency response of the LTI system as defined in Eq. (3.85). Equation (3.88) thus relates the input and the output of an LTI system in the frequency domain.

From Eq. (3.88), we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \quad (3.89)$$

Thus, the frequency response of an LTI discrete-time system is given by the ratio of the Fourier transform $Y(e^{j\omega})$ of the output sequence $y[n]$ to the Fourier transform $X(e^{j\omega})$ of the input sequence $x[n]$.

It follows from the input-output relation of Eq. (3.88) of an LTI discrete-time system in the frequency domain that the output cannot contain sinusoidal components of frequencies that are not present in the input and the system. As a result, if the output of a system has new frequency components, then the system is either nonlinear or time-varying or both (Problem 3.50).

Frequency Response of LTI FIR Discrete-Time Systems

The LTI FIR discrete-time systems are characterized by an input–output relation of the form of Eq. (2.118) and repeated below for convenience:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k], \quad N_1 < N_2.$$

Applying the discrete-time Fourier transform (DTFT) to the above equation and making use of the linearity and the time-shifting properties of Table 3.4, we arrive at the input–output relation of the LTI system in the transform-domain given by

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k} X(e^{j\omega}), \quad (3.92)$$

where $Y(e^{j\omega})$ and $X(e^{j\omega})$ are the Fourier transforms of the sequences $y[n]$ and $x[n]$, respectively. In developing Eq. (3.92), it has been tacitly assumed that $Y(e^{j\omega})$ and $X(e^{j\omega})$ exist. From the above equation, we arrive at the expression for its frequency response $H(e^{j\omega})$ as given below:

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}, \quad (3.93)$$

which is seen to be a polynomial in $e^{j\omega}$.

Frequency Response of LTI IIR Discrete-Time Systems

The LTI IIR discrete-time systems we shall be concerned with in this book are characterized by linear constant coefficient difference equations of the form of Eq. (2.90) and repeated below for convenience:

$$\sum_{k=0}^N d_k y[n-k] = \sum_{k=0}^M p_k x[n-k].$$

Applying the discrete-time Fourier transform (DTFT) to the above equation and making use of the linearity and the time-shifting properties of Table 3.4, we arrive at the input-output relation of the LTI system in the transform-domain given by

$$\sum_{k=0}^N d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^M p_k e^{-j\omega k} X(e^{j\omega}). \quad (3.94)$$

The above equation can be alternately written as

$$\left(\sum_{k=0}^N d_k e^{-j\omega k} \right) Y(e^{j\omega}) = \left(\sum_{k=0}^M p_k e^{-j\omega k} \right) X(e^{j\omega}). \quad (3.95)$$

Thus, from Eq. (3.95), the expression for its frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M p_k e^{-j\omega k}}{\sum_{k=0}^N d_k e^{-j\omega k}}, \quad (3.96)$$

which is a rational function in $e^{j\omega}$.