EENG 479 : Digital Signal Processing (DSP)

Lecture #6: Chapter 3 & 5 : DTFT

Prof. Mohab A. Mangoud

Professor of Wireless Communications (Networks, IoT and AI)
University of Bahrain, College of Engineering
Department of Electrical and Electronics Engineering
P.O.Box 32038- Kingdom of Bahrain
mmangoud@uob.edu.bh
http://mangoud.com

Chapter (3)
Continuous time Fourier Transform
Discrete time Fourier Transform

In Section 2.2.3, we pointed out that any arbitrary sequence can be represented in the time domain as a weighted linear combination of delayed unit sample sequences $\{\delta[n-k]\}$. An important consequence of this representation, derived in Section 2.5.1, is the input–output characterization of a linear, time-invariant (LTI) digital filter in the time domain by means of the convolution sum describing the output sequence in terms of a weighted linear combination of its delayed impulse responses. In many applications, it is convenient to consider an alternate description of a sequence in terms of complex exponential sequences of the form $\{e^{-j\omega n}\}$, where ω is the normalized frequency variable in radians. This leads to a particularly useful representation of discrete-time sequences and LTI discrete-time systems in the frequency domain.[1]

The frequency-domain representation of a discrete-time sequence discussed in this chapter is the discrete-time Fourier transform by which a time-domain sequence is mapped into a continuous function of the frequency variable ω. Because of the periodicity of the discrete-time Fourier transform, the corresponding discrete-time sequence can be simply obtained by computing its Fourier series representation. Since the representation is in terms of an infinite series, existence of the discrete-time Fourier transform is examined along with its properties.

The frequency-domain representation of a linear, time-invariant (LTI) discrete-time system is its frequency response given by the discrete-time Fourier transform of its impulse response.

Four Fourier Representations

Time Domain	Periodic	Non-periodic	
Discrete	**Discrete-Time Fourier Series** $$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi kn/N}$$ $$\tilde{x}[n] = \sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N}$$	**Discrete-Time Fourier Transform** $$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$ $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$$	**Periodic**
Continuous	**Fourier Series** $$X[k] = \frac{1}{T}\int_{-T/2}^{T/2}\tilde{x}(t)e^{-j2\pi kt/T}\,dt$$ $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty}X[k]e^{j2\pi kt/T}$$	**Fourier Transform** $$X(j\omega) = \int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$$ $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\,d\omega$$	**Non-periodic**
	Discrete	**Continuous**	Freq. Domain

The Continuous-Time Fourier Transform

Probably the easiest way to introduce the discrete-time Fourier transform (DTFT) is through its counterpart, the continuous-time Fourier transform (CTFT). As you will recall, the CTFT and its inverse can be expressed as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,dt \tag{1}$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \tag{2}$$

$$x_a(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega).$$

Note that we use the upper case variable Ω to indicate continuous-time angular frequency in this course. As we have discused, the first equation (which is actually the inverse CTFT) expresses the fact that a finite-energy time function $x(t)$ can be represented as a weighted linear combination of complex exponentials $e^{j\Omega t}$. The second equation (the actual CTFT) tells you how to compute the weights $X(j\Omega)$. Note that the frequencies Ω include all real numbers.

general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$. It can be expressed in polar form as

$$X_a(j\Omega) = |X_a(j\Omega)|e^{j\theta_a(\Omega)},$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}.$$

The quantity $|X_a(j\Omega)|$ is called the *magnitude spectrum,* and the quantity $\theta_a(\Omega)$ is called the *phase spectrum,* with both spectrums being real functions of Ω.

Condition for CTFT

In general, the CTFT $X_a(j\Omega)$ defined by Eq. (3.1) exists if the continuous-time signal $x_a(t)$ satisfies the *Dirichlet conditions:*

(a) The signal has a finite number of finite discontinuities and a finite number of maxima and minima in any finite interval.

(b) The signal is absolutely integrable; that is,
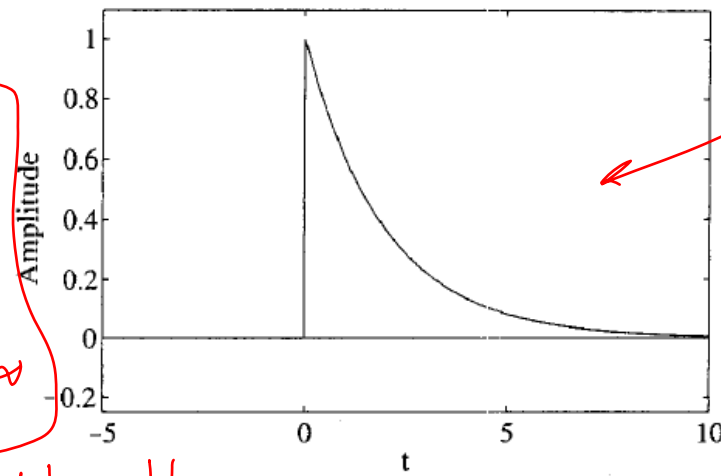
$$\int_{-\infty}^{\infty} |x_a(t)|dt < \infty. \tag{3.3}$$

We illustrate the CTFT computation in Example 3.1.

$$x_a(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |x_a(t)| \, dt = \int_{0}^{\infty} e^{-\alpha t} \, dt$$

$$= -\frac{e^{-\alpha t}}{\alpha} \Big|_{0}^{\infty} = \frac{1}{\alpha} < \infty$$

so $x_a(t)$ is absolutly integrable

$$X_a(j\Omega) = \int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega t} \, dt$$

$$= \frac{-1}{\alpha + j\Omega} e^{-(\alpha + j\Omega)t} \Big|_{0}^{\infty}$$

$$X_a(j\Omega) = \frac{1}{\alpha + j\Omega}$$

Figure 3.1: Plot of the continuous-time function of Eq. (3.4) for $\alpha = 0.5$.
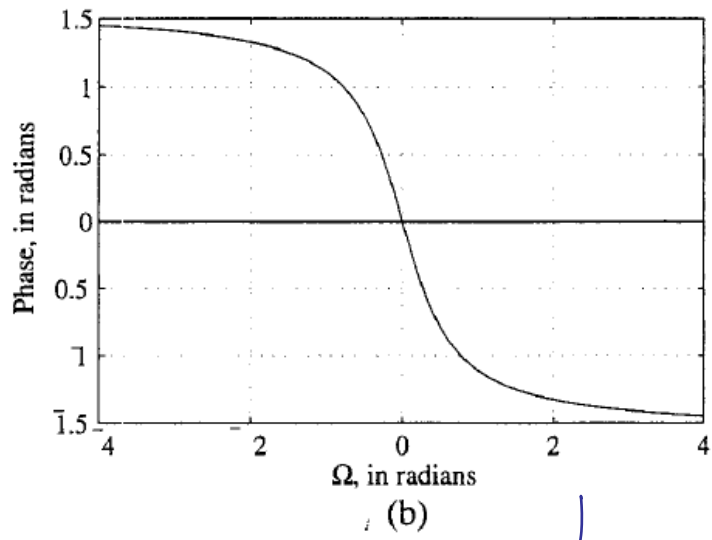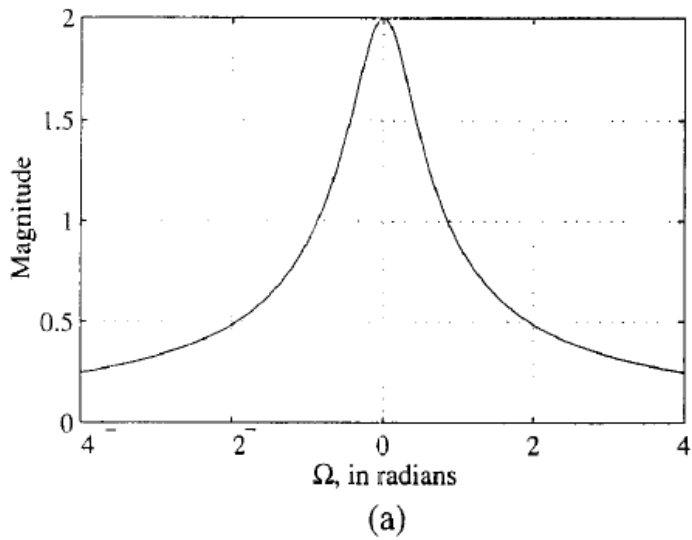

(a)

(b)

Figure 3.2: (a) Magnitude and (b) phase of $X_a(j\Omega) = 1/(0.5 + j\Omega)$.

$$\frac{1}{\sqrt{\alpha^2 + \Omega^2}} e^{-j \tan^{-1}\left(\frac{\Omega}{\alpha}\right)}$$

3.1.2 Energy Density Spectrum

The total energy $\mathcal{E}_x$ of a finite-energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) \, dt. \tag{3.7}$$

The energy can also be expressed in terms of the CTFT $X_a(j\Omega)$. To this end, we first replace $x_a^*(t)$ in the above equation by its inverse CTFT expression as shown below:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} \, d\Omega \right] dt.$$

Interchanging the order of the integrations in the above, we get

$$\mathcal{E}_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} \, dt \right] d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) \, d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega. \tag{3.8}$$

Combining Eqs. (3.7) and (3.8), we arrive at

$$\int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega, \tag{3.9}$$

which is more commonly known as the *Parseval's relation* for finite-energy continuous-time signals.

The integrand $|X_a(j\Omega)|^2$ on the right-hand side of Eq. (3.8) is called the *energy density spectrum* of the continuous-time signal $x_a(t)$ and is usually denoted by the symbol $S_{xx}(\Omega)$; that is,

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2.$$

The energy $\mathcal{E}_{x,r}$ over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ of the signal $x_a(t)$ can be computed by integrating $S_{xx}(\Omega)$ over this range:

$$\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\, d\Omega.$$

3.1.3 Band-limited Continuous-Time Signals

A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$, whereas a *band-limited* continuous-time signal has a spectrum that is limited to a portion of the above frequency range. An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$; that is,

$$X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_a, \\ 0, & \Omega_b < |\Omega| < \infty. \end{cases}$$

However, an ideal band-limited signal cannot be generated in practice, and, for practical purposes, it is sufficient to ensure that for a band-limited signal outside the specified frequency range, the signal energy is arbitrarily small.

Band-limited signals are classified according to the frequency range where most of the signal's energy is concentrated. A *lowpass* continuous-time signal has a spectrum occupying the frequency range $0 \leq |\Omega| \leq \Omega_p < \infty$, where Ω_p is called the *bandwidth* of the signal. Likewise, a *highpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \leq |\Omega| < \infty$, where the *bandwidth* of the signal is from Ω_p to ∞. Finally, a *bandpass* continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \leq |\Omega| \leq \Omega_H < \infty$, where $\Omega_H - \Omega_L$ is its *bandwidth*. A precise definition of the bandwidth depends on applications. As can be seen from Figure 3.2(a), the continuous-time signal of Eq. (3.4) is a lowpass signal. It can be shown that 80% of the energy of this signal is contained in the frequency range $0 \leq |\Omega| \leq 0.4898\pi$, and hence, we can define the 80% bandwidth of the signal to be 0.4898π radians (Problem 3.10).

3.1.4 The Frequency Response of an LTI Continuous-Time System

As indicated by Eq. (1.2), the output response $y_a(t)$ of an initially relaxed linear, time-invariant continuous-time system characterized by an impulse response $h_a(t)$ for an input signal $x_a(t)$ is given by the convolution integral

$$y_a(t) = \int_{-\infty}^{\infty} h_a(t-\tau)x_a(\tau)d\tau. \tag{3.10}$$

$$y_a(t) = h_a(t) \circledast x_a(t)$$

$$\begin{cases} y_n \overset{LTI}{=} x_n \circledast h_n \\ y_n = \sum_{k=-\infty}^{\infty} h_n x_{n-k} \end{cases} \tag{3.10}$$

$$\longrightarrow \textit{discrete}$$

Applying the CTFT to both sides of Eq. (3.10), we have

$$Y_a(j\Omega) = H_a(j\Omega)X_a(j\Omega), \tag{3.11}$$

where $Y_a(j\Omega)$, $X_a(j\Omega)$ and $H_a(j\Omega)$ are, respectively, the CTFTs of $y_a(t)$, $x_a(t)$, and $h_a(t)$. The function $H_a(j\Omega)$ is called the *frequency response* of the LTI continuous-time system.

3.2 The Discrete-Time Fourier Transform

The *discrete-time Fourier transform* (DTFT) of a discrete-time sequence $x[n]$ is a representation of the sequence in terms of the complex exponential sequence $\{e^{j\omega n}\}$, where ω is the real frequency variable. If there is no ambiguity, for brevity often the discrete-time Fourier transform is called simply the *Fourier transform* (FT). The Fourier transform representation of a sequence, if it exists, is unique, and the original sequence can be computed from its transform representation by an inverse transform operation. We first define the forward transform and derive its inverse transform. We then describe the condition for its existence and review its important properties.

The *discrete-time Fourier transform* $X(e^{j\omega})$ of a sequence $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

$\mathcal{F}\{x[n]\}$

the *analysis equation,*

$$\mathcal{F}^{-1}\{X(e^{j\omega})\}$$

synthesis equation,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} \, d\omega, \tag{3.16}$$

called the *inverse discrete-time Fourier transform.* It should be noted that even though the integration in Eq. (3.16) can be carried out over any interval of duration 2π, it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.16) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π.

$$\mathcal{F}\{x[n]\}$$

to denote the $X(e^{j\omega})$ of the sequence $x[n]$. Likewise, we shall use the operator symbol

$$\mathcal{F}^{-1}\{X(e^{j\omega})\}$$

to denote the inverse Fourier transform $x[n]$ of the transform $X(e^{j\omega})$. A discrete-time Fourier transform pair will be denoted as

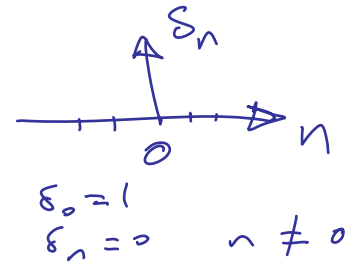$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}). \tag{3.17}$$

Example ①

$$X_n = \delta_n \qquad X(e^{j\omega}) = ??$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X_n e^{-j\omega n}$$

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta_n e^{-j\omega n} = 1$$

$$\therefore \qquad \delta_n \xRightarrow{\text{DTFT}} 1$$

$\delta_0 = 1$

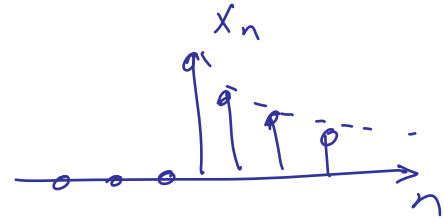$\delta_n = 0 \qquad n \neq 0$

Example ②

$$X_n = \alpha^n u(n) \qquad |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}}$$

As can be seen from the definition, the discrete-time Fourier transform $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω. However, unlike the continuous-time Fourier transform, it is a periodic function in ω with a period 2π. To verify this latter property, observe that for any integer k,

periodicity property of the transform.

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi kn}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}), \quad \text{for all values of } k,$$

where we have used the fact $e^{-j2\pi kn} = 1$. It therefore follows that Eq. (3.12) represents the Fourier series expansion of the periodic function $X(e^{j\omega})$. As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} \, d\omega, \qquad (3.16)$$

IDFT

called the *inverse discrete-time Fourier transform.* It should be noted that even though the integration in Eq. (3.16) can be carried out over any interval of duration 2π, it is a common practice to choose the interval $[-\pi, \pi]$. The inverse discrete-time Fourier transform given by Eq. (3.16) can be interpreted as a linear combination of infinitesimally small complex exponential signals of the form $\frac{1}{2\pi}e^{j\omega n}d\omega$, weighted by the complex constant $X(e^{j\omega})$ over the angular frequency range from $-\pi$ to π.

Proof of IDFT equation

To verify that the integral on the right-hand side of Eq. (3.16) indeed results in the inverse FT $x[n]$, we substitute the expression for $X(e^{j\omega})$ from Eq. (3.12) in Eq. (3.16), arriving at

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\omega\ell} \right) e^{j\omega n}\, d\omega.$$

The order of integration and the summation on the right-hand side of the above equation can be interchanged if the summation inside the brackets converges uniformly, that is, if $X(e^{j\omega})$ exists. Under this condition, we get from the above

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-\ell)}\, d\omega \right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\, \frac{\sin \pi(n-\ell)}{\pi(n-\ell)}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\, \mathrm{sinc}(n-\ell).$$

For $n \neq \ell$, $\sin \pi(n-\ell) = 0$, and as a result, $\mathrm{sinc}(n-\ell) = 0$. For $n = \ell$, $\mathrm{sinc}(n-\ell) = 0/0$. In this case, using L'Hospital rule, we get

$$\lim_{n \to \ell} \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \frac{\pi \cos(\pi n)}{\pi} = 1.$$

Thus,

$$\mathrm{sinc}(n-\ell) = \begin{cases} 1, & n = \ell, \\ 0, & n \neq \ell, \end{cases}$$

$$= \delta[n-\ell]. \qquad (3.18)$$

Hence,

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\, \mathrm{sinc}(n-\ell) = \sum_{\ell=-\infty}^{\infty} x[\ell]\, \delta[n-\ell] = x[n],$$

using the sampling property of the unit sample sequence.

3.2.2 Basic Properties

$X(e^{j\omega})$ is a complex function of the real variable ω

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}),$$

$$\begin{cases} X_{re}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}, \\ X_{im}(e^{j\omega}) = \frac{1}{2j}\{X(e^{j\omega}) - X^*(e^{j\omega})\}, \end{cases}$$

the polar form

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j[\theta(\omega)+2\pi k]} = |X(e^{j\omega})|e^{j\theta(\omega)},$$

magnitude function,   *phase function,*

the *Fourier spectrum* and likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are referred to as the *magnitude spectrum* and *phase spectrum,* respectively.

The relations between the rectangular and polar forms

$$X_{re}(e^{j\omega}) = |X(e^{j\omega})|\cos\ \theta(\omega),$$

$$X_{im}(e^{j\omega}) = |X(e^{j\omega})|\sin\ \theta(\omega),$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega}) = X_{re}^2(e^{j\omega}) + X_{im}^2(e^{j\omega}),$$

$$\tan\theta(\omega) = \frac{X_{im}(e^{j\omega})}{X_{re}(e^{j\omega})}.$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)},$$

$$\theta(\omega) = \arg\{X(e^{j\omega})\}.$$

3.2.3 Symmetry Relations

$$x[n] = x_{re}[n] + jx_{im}[n]$$

Table 3.1: Symmetry relations of the discrete-time Fourier transform of a complex sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$ $\longleftrightarrow$	$X(e^{-j\omega})$
$x^*[-n]$ $\longrightarrow$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

For real sequence, $x_{im}[n] = 0$. Hence, from Eq. (3.33), $X_{ca}(e^{j\omega}) = 0$, and thus, from Eq. (3.31b), we have $X(e^{j\omega}) = X^*(e^{j\omega})$, implying that $X(e^{j\omega})$ is a conjugate-symmetric function. As a result, the real part $X_{re}(e^{j\omega})$ and imaginary part $X_{im}(e^{j\omega})$ of the Fourier transform of a real sequence are, respectively, even and odd functions of ω.

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
Symmetry relations	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$|X(e^{j\omega})| = |X(e^{-j\omega})|$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

3.2.4 Convergence Condition

Now, an infinite series of the form of Eq. (3.12) may or may not converge. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is said to exist if the series in Eq. (3.12) converges in some sense. Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}, \tag{3.42}$$

denote the partial sum of the weighted complex exponentials in Eq. (3.12). Then for *uniform convergence* of $X(e^{j\omega})$,

$$\lim_{K \to \infty} X_K(e^{j\omega}) = X(e^{j\omega}).$$

Now, if $x[n]$ is an *absolutely summable sequence*, that is, if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty, \tag{3.43}$$

then

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \left| e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

for all values of ω guaranteeing the existence of $X(e^{j\omega})$. Thus, Eq. (3.43) is a sufficient condition for the existence of the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$. Moreover, it can be shown that for an absolutely summable sequence, the infinite series of Eq. (3.12) defining the Fourier transform converges uniformly for all values of ω.

A large class of sequences encountered in practice are of finite length with finite sample values. These sequences are absolutely summable, and hence, their Fourier transforms converge uniformly. On the other hand, infinite length sequences may or may not converge uniformly. The sequence $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$, of Example 3.5 is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1 - |\alpha|} < \infty,$$
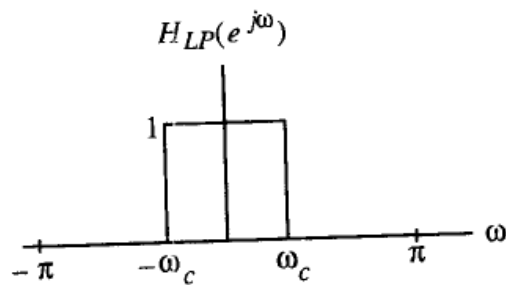
Figure 3.5: Plot of the Fourier transform of Eq. (3.45).

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{LP}[n] = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0, \\ \dfrac{\sin \omega_c n}{\pi n}, & n \ne 0. \end{cases}$$

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad (-\infty < n < \infty)$$

The mean-square convergence property of the sequence $h_{LP}[n]$ discussed in Example 3.8 can be further illustrated by examining the plot of the function

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}, \qquad H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}, \qquad (3.50)$$

for various values of K, as shown in Figure 3.6. It can be seen from this figure that, independent of the number of terms in the above sum, there are ripples in the plot of $H_{LP}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$. The number of ripples increases as K increases, with the height of the largest ripple remaining the same for all values of K. As K goes to infinity, the condition of Eq. (3.44) holds, indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$. The oscillatory behavior in the plot of $H_{LP,K}(e^{j\omega})$ approximating a Fourier transform $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity, as indicated in Figure 3.6, is commonly known as the *Gibbs phenomenon.* We shall return to this phenomenon in the design of FIR filters based on the windowed Fourier series discussed in Section 10.2.3.
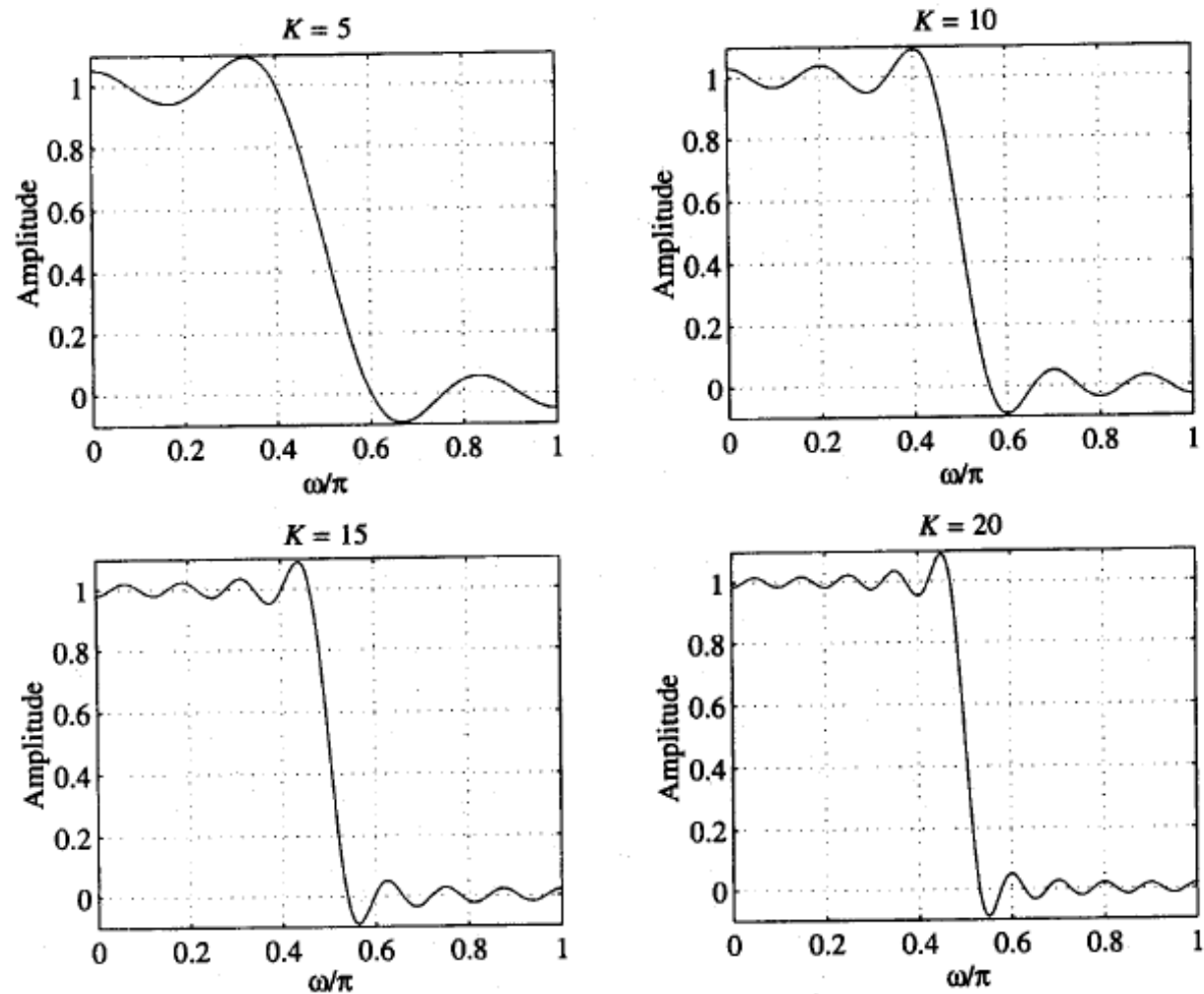
Figure 3.6: Frequency response plots of Eq. (3.51) for various values of K.

Existence of the Fourier Representations

Under what conditions do the Fourier representations exist?

- The DTFS and its inverse always exist for any discrete-time periodic signal (after all, they are just finite sums).

- However, the other Fourier representations involve infinite sums or integrals, so we need to be careful with *convergence*.

- As an example, consider the convergence properties of the DTFT.

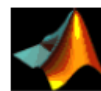- The DTFT converges uniformly if the signal is *absolutely summable*:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

- On the other hand, the DTFT only converges in mean square if the signal is only *square summable*:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$
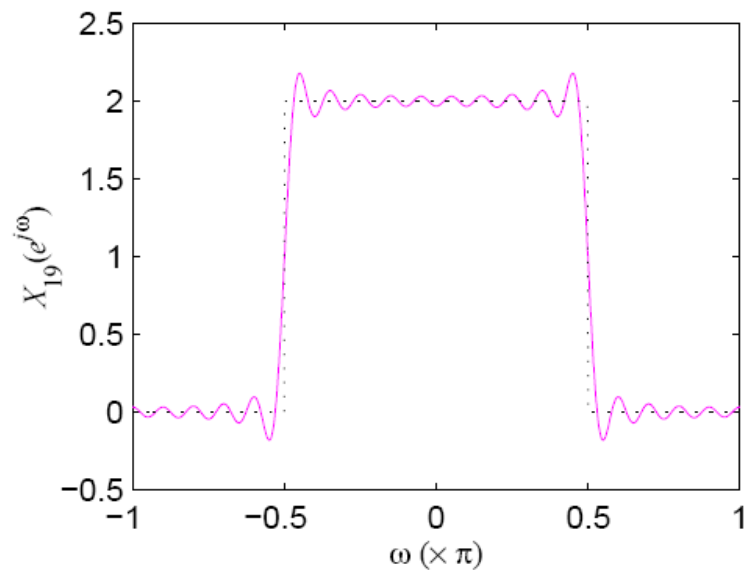
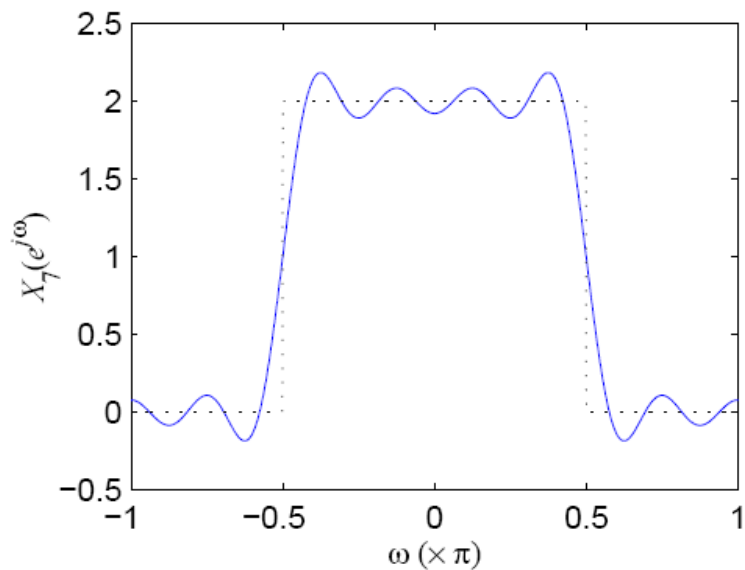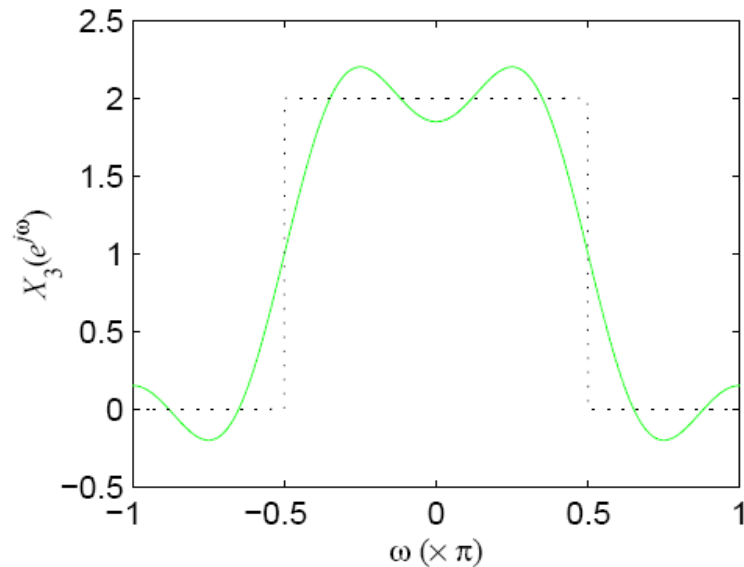- This leads to Gibbs' effect, now demonstrated for a sinc pulse.
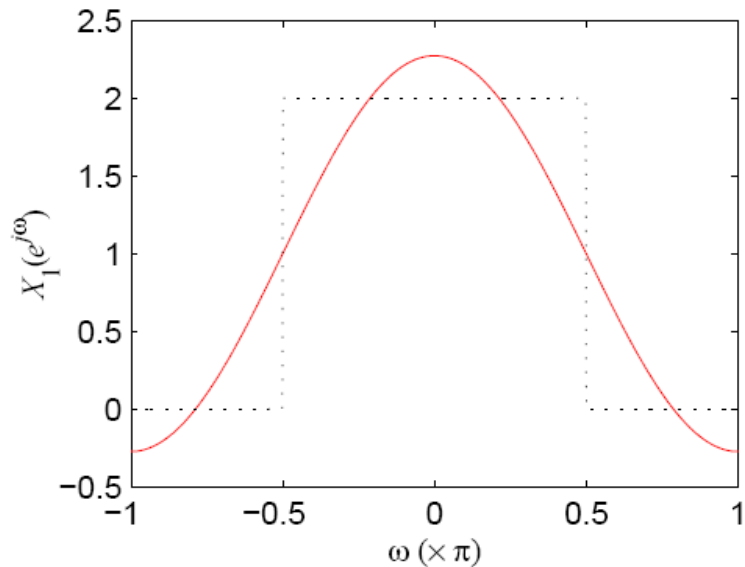
Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \quad (-\infty < n < \infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\displaystyle\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (|\alpha| < 1)$	$\displaystyle\frac{1}{1 - \alpha e^{-j\omega}}$
$(n + 1)\alpha^n \mu[n], \quad (|\alpha| < 1)$	$\displaystyle\frac{1}{(1 - \alpha e^{-j\omega})^2}$
$h_{LP}[n] = \dfrac{\sin \omega_c n}{\pi n}, \quad (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

3.3 Discrete-Time Fourier Transform Theorems

There are a number of important theorems of the discrete-time Fourier transform that are useful in digital signal processing applications. These theorems can be used to determine the Fourier transforms of sequences obtained by combining sequences with known transforms. We review these theorems in this section. For compactness, the theorems are stated using the operator notation introduced in Eq. (3.17), and we shall make use of the following Fourier transform pairs:

$$g[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega}), \tag{3.56a}$$

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}). \tag{3.56b}$$

The proofs of most of the theorems given here are quite straightforward and are left as exercises.

Convolution Theorem

The Fourier transform $Y(e^{j\omega})$ of the convolution sum of two sequences, $y[n] = g[n] \circledast h[n]$, is given by the product of their Fourier transforms $G(e^{j\omega})H(e^{j\omega})$; that is,

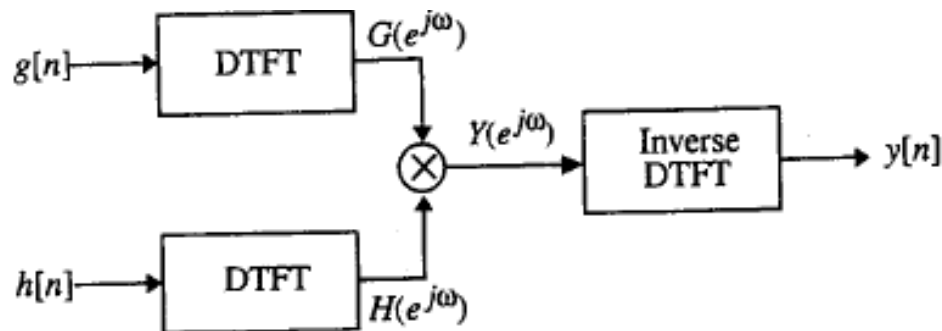$$g[n] \circledast h[n] \xrightarrow{\mathcal{F}} G(e^{j\omega})H(e^{j\omega}). \tag{3.65}$$

Figure 3.9: Linear convolution using the discrete-time Fourier transform.

Modulation Theorem

The Fourier transform $Y(e^{j\omega})$ of the product of two sequences given by $y[n] = g[n]h[n]$ is given by the convolution integral of their Fourier transforms $\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) \, d\theta$; that is,

$$g[n]h[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) \, d\theta. \tag{3.66}$$

The modulation theorem plays a key role in the amplitude modulation scheme used in digital communications. The modulation theorem is also known as the *windowing theorem*. One application of this theorem, to be considered later in Section 10.2, is in the design of linear-phase FIR filter based on the windowing of the impulse response of an ideal linear-phase filter with a doubly-infinite impulse response.

Parseval's Relation

This theorem expresses the sum of sample-by-sample product of two complex sequences in terms of an integral of the product of their Fourier transforms. Specifically, the most general form of this theorem is given by

$$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega. \tag{3.67}$$

Linearity Theorem

Consider a sequence $x[n] = \alpha g[n] + \beta h[n]$ obtained by a linear combination of $g[n]$ and $h[n]$, where α and β are arbitrary constants. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is then given by $\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$; that is,

$$\alpha g[n] + \beta h[n] \xleftrightarrow{\mathcal{F}} \alpha G(e^{j\omega}) + \beta H(e^{j\omega}). \tag{3.57}$$

It should be noted that the Fourier transforms of the conjugate-symmetric and conjugate-antisymmetric parts of a sequence derived in Eqs. (3.34) and (3.35) made use of the linearity theorem.

Time-Reversal Theorem

The Fourier transform of the time-reversed sequence $g[-n]$ is given by $G(e^{-j\omega})$; that is,

$$g[-n] \xleftrightarrow{\mathcal{F}} G(e^{-j\omega}). \tag{3.58}$$

Time-Shifting Theorem

The Fourier transform of the delayed sequence $x[n] = g[n - n_o]$, with n_o an integer, is given by $X(e^{j\omega}) = e^{-j\omega n_o} G(e^{j\omega})$; that is,

$$g[n - n_o] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_o} G(e^{j\omega}). \tag{3.59}$$

It follows from Equation 3.59 that since $|e^{-j\omega n_o}| = 1$, $|G(e^{j\omega})| = |X(e^{j\omega})|$; that is, the magnitude spectrum is unchanged by shifting a signal in time.

Frequency-Shifting Theorem

The Fourier transform of a sequence $x[n] = e^{j\omega_o n} g[n]$ is given by $X(e^{j\omega}) = G(e^{j(\omega - \omega_o)})$; that is,

$$e^{j\omega_o n} g[n] \xleftrightarrow{\mathcal{F}} G(e^{j(\omega - \omega_o)}). \tag{3.63}$$

Differentiation-in-Frequency Theorem

The Fourier transform of a sequence $x[n] = n g[n]$ is given by $X(e^{j\omega}) = j\frac{dG(e^{j\omega})}{d\omega}$; that is,

$$n g[n] \xleftrightarrow{\mathcal{F}} j\frac{dG(e^{j\omega})}{d\omega}. \tag{3.64}$$

Table 3.4: Discrete-time Fourier transform theorems.

Theorem	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-reversal	$g[-n]$	$G(e^{-j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega - \omega_o)}\right)$
Differentiation-in-frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's Relation	$\displaystyle\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

Properties of the Discrete-Time Fourier Transform

Property	Time domain	Frequency domain
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Differentiation (frequency)	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
Time-shift	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-shift	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega - \omega_0)})$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$
Time-reversal	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Symmetry (real)	$\Im\{x[n]\} = 0$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
Symmetry (imag)	$\Re\{x[n]\} = 0$	$X(e^{j\omega}) = -X^*(e^{-j\omega})$
Parseval	$\displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega$	

3.6 DTFT Computation Using MATLAB

The *Signal Processing Toolbox* in MATLAB includes a number of M-files to aid in the DTFT-based analysis of discrete-time signals. Specifically, the functions that can be used are `freqz`, `abs`, `angle`, and `unwrap`. In addition, the built-in MATLAB functions `real` and `imag` are also useful in some applications.

FREQZ Digital filter frequency response.
  [H,W] = FREQZ(B,A,N) returns the N-point complex frequency response
  vector H and the N-point frequency vector W in radians/sample of
  the filter:

```
         jw            -jw           -jmw
   jw  B(e)    b(1) + b(2)e + .... + b(m+1)e
  H(e) = ---- = -------------------------------------
         jw            -jw           -jnw
     A(e)    a(1) + a(2)e + .... + a(n+1)e
```

given numerator and denominator coefficients in vectors B and A.
The frequency response is evaluated at N points equally spaced around the upper half of the unit circle.
 If N isn't specified, it defaults to 512.

In this book, many of the Fourier transforms we shall encounter are rational functions in $e^{-j\omega}$, that is, ratios of polynomials in $e^{-j\omega}$, and are of the form

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{D(e^{j\omega})} = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}. \tag{3.55}$$

DTFT Computation Using MATLAB

- For example, the statement

  ```
  H = freqz(num,den,w)
  ```

  returns the frequency response values as a vector H of a DTFT defined in terms of the vectors num and den containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector w

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```matlab
% Program 3_1
% Discrete-Time Fourier Transform Computation
%
% Read in the desired number of frequency samples
k = input('Number of frequency points = ');
% Read in the numerator and denominator coefficients
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
% Compute the frequency response
w = 0:pi/(k-1):pi;
h = freqz(num, den, w);

% Plot the frequency response
subplot(2,2,1)
plot(w/pi,real(h));grid
title('Real part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,2)
plot(w/pi,imag(h));grid
title('Imaginary part')
xlabel('\omega/\pi'); ylabel('Amplitude')
subplot(2,2,3)
plot(w/pi,abs(h));grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')
subplot(2,2,4)
plot(w/pi,angle(h));grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')
```
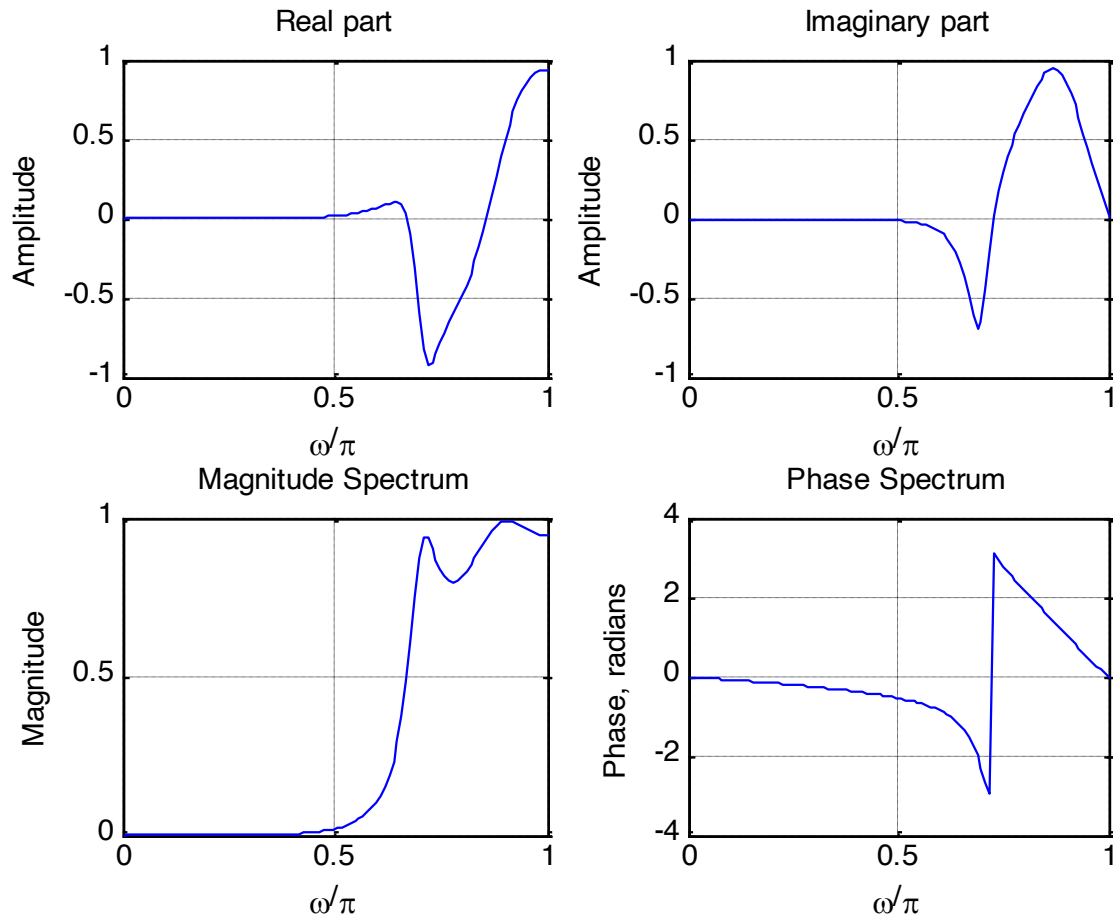
Number of frequency points = 100
Numerator coefficients = [0.008 -0.033 0.05 -0.033 0.008]
Denominator coefficients = [1 2.37 2.7 1.6 0.41]

The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range

- Thus. the phase functions of some sequences exhibit discontinuities of 2π radians in the plot

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The Unwrapped Phase Function

- For example, there is a discontinuity of 2π at $\omega = 0.72$ in the phase response below

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$


Phase Spectrum

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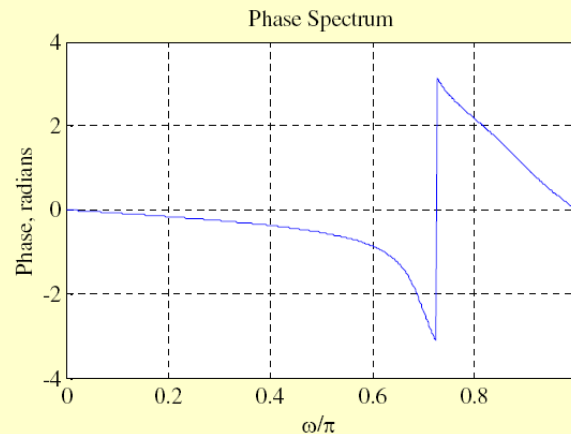
The Unwrapped Phase Function

- In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π

- Process of discontinuity removal is called unwrapping the phase

- The unwrapped phase function will be denoted as $\theta_c(\omega)$
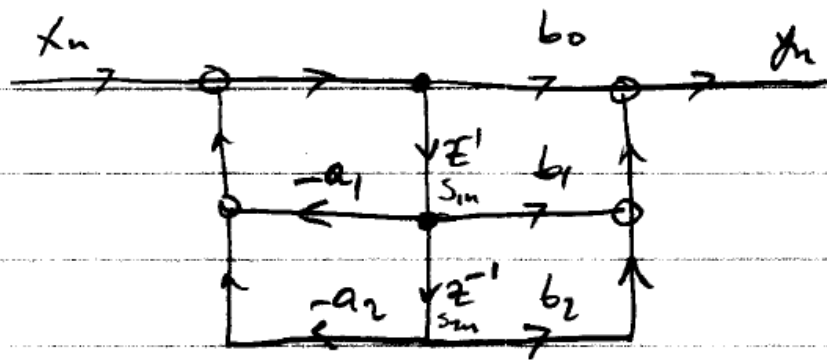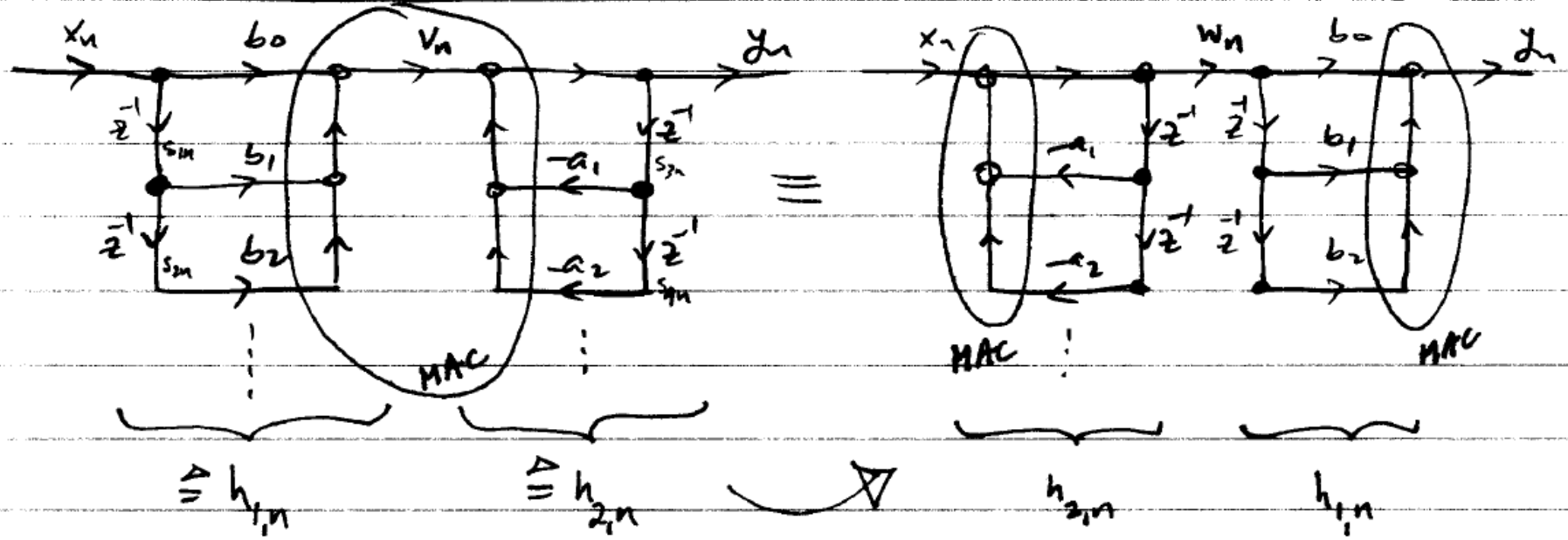
25

The Unwrapped Phase Function

- In MATLAB, the unwrapping can be implemented using the M-file `unwrap`

- The unwrapped phase function of the DTFT of previous page is shown below

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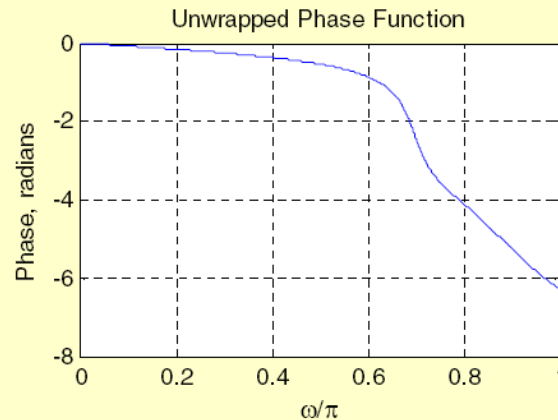
$\Rightarrow$ identical I/O behavior

recall: fundamental direct SFG

"Direct Form II [OS]"

3.8 The Frequency Response of an LTI Discrete-Time System

Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies. Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property of the system. Since a sinusoidal signal can be expressed in terms of an exponential signal, the response of the LTI system to an exponential input is of practical interest. This leads to the concept of frequency response, a transform-domain representation of the LTI discrete-time system. We first define the frequency response, investigate its properties, and describe some of its applications. The computation of the time-domain representation of the LTI system from its frequency response is outlined.
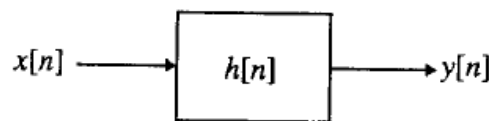
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Figure 3.12: An LTI discrete-time system.

3.8.1 Definition

An important property of an LTI system is that for certain types of input signals, called *eigenfunctions*, the output signal is the input signal multiplied by a complex constant. We consider here one such eigenfunction as the input. Recall from Section 2.5.1, the input–output relationship of an LTI discrete-time system as shown in Figure 3.12, with an impulse response $h[n]$, is given by the convolution sum of Eq. (2.73b) and is of the form

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k], \tag{3.81}$$

where $y[n]$ and $x[n]$ are, respectively, the output and the input sequences. Now, if the input $x[n]$ is a complex exponential sequence of the form

$$x[n] = e^{j\omega n}, \qquad -\infty < n < \infty, \tag{3.82}$$

then, from Eq. (3.81), the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}, \qquad (3.83)$$

which can be rewritten as

$$y[n] = H(e^{j\omega})e^{j\omega n}, \qquad (3.84)$$

where we have used the notation

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}. \qquad (3.85)$$

It can be seen from Eq. (3.85) that for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$. Thus, $e^{j\omega n}$ is an eigenfunction of the system.

The quantity $H(e^{j\omega})$ defined above is called the *frequency response* of the LTI discrete-time system, and it provides a frequency-domain description of the system. Note from Eq. (3.85) that $H(e^{j\omega})$ is precisely the Fourier transform of the impulse response $h[n]$ of the system.

Just like any other discrete-time Fourier transform, in general, $H(e^{j\omega})$ is also a complex function of ω with a period 2π and can be expressed in terms of its real and imaginary parts or its magnitude and phase. Thus,

$$H(e^{j\omega}) = H_{\text{re}}(e^{j\omega}) + j H_{\text{im}}(e^{j\omega})$$

$$= |H(e^{j\omega})|e^{j\theta(\omega)},$$

where $H_{\text{re}}(e^{j\omega})$ and $H_{\text{im}}(e^{j\omega})$ are, respectively, the real and imaginary parts of $H(e^{j\omega})$, and

$$\theta(\omega) = \arg\{H(e^{j\omega})\}. \tag{3.86}$$

The quantity $|H(e^{j\omega})|$ is called the *magnitude response* and the quantity $\theta(\omega)$ is called the *phase response* of the LTI discrete-time system. Design specifications for the discrete-time systems, in many applications, are given in terms of the magnitude response or the phase response or both. In some cases, the magnitude function is specified in decibels as defined below:

$$\mathcal{G}(\omega) = 20 \log_{10} |H(e^{j\omega})| \quad \text{dB}, \tag{3.87}$$

where $\mathcal{G}(\omega)$ is called the *gain function*. The negative of the gain function, $\mathcal{A}(\omega) = -\mathcal{G}(\omega)$, is called the *attenuation* or *loss function*.

3.6.2 Frequency-Domain Characterization of the LTI Discrete-Time System

We now derive the frequency-domain representation of an LTI discrete-time system. If $Y(e^{j\omega})$ and $X(e^{j\omega})$ denote the Fourier transforms of the output and input sequences, $y[n]$ and $x[n]$, respectively, then applying the convolution theorem of Table 3.4 to Eq. (3.81), we arrive at

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

(3.88)

where $H(e^{j\omega})$ is the frequency response of the LTI system as defined in Eq. (3.85). Equation (3.88) thus relates the input and the output of an LTI system in the frequency domain.

From Eq. (3.88), we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

(3.89)

Thus, the frequency response of an LTI discrete-time system is given by the ratio of the Fourier transform $Y(e^{j\omega})$ of the output sequence $y[n]$ to the Fourier transform $X(e^{j\omega})$ of the input sequence $x[n]$.

It follows from the input–output relation of Eq. (3.88) of an LTI discrete-time system in the frequency domain that the output cannot contain sinusoidal components of frequencies that are not present in the input and the system. As a result, if the output of a system has new frequency components, then the system is either nonlinear or time-varying or both (Problem 3.50).

Frequency Response of LTI FIR Discrete-Time Systems

The LTI FIR discrete-time systems are characterized by an input–output relation of the form of Eq. (2.118) and repeated below for convenience:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k], \qquad N_1 < N_2.$$

Applying the discrete-time Fourier transform (DTFT) to the above equation and making use of the linearity and the time-shifting properties of Table 3.4, we arrive at the input–output relation of the LTI system in the transform-domain given by

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}X(e^{j\omega}), \qquad (3.92)$$

where $Y(e^{j\omega})$ and $X(e^{j\omega})$ are the Fourier transforms of the sequences $y[n]$ and $x[n]$, respectively. In developing Eq. (3.92), it has been tacitly assumed that $Y(e^{j\omega})$ and $X(e^{j\omega})$ exist. From the above equation, we arrive at the expression for its frequency response $H(e^{j\omega})$ as given below:

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}, \qquad (3.93)$$

which is seen to be a polynomial in $e^{j\omega}$.

Frequency Response of LTI IIR Discrete-Time Systems

The LTI IIR discrete-time systems we shall be concerned with in this book are characterized by linear constant coefficient difference equations of the form of Eq. (2.90) and repeated below for convenience:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k].$$

Applying the discrete-time Fourier transform (DTFT) to the above equation and making use of the linearity and the time-shifting properties of Table 3.4, we arrive at the input–output relation of the LTI system in the transform-domain given by

$$\sum_{k=0}^{N} d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} p_k e^{-j\omega k} X(e^{j\omega}). \tag{3.94}$$

The above equation can be alternately written as

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k} \right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k} \right) X(e^{j\omega}). \tag{3.95}$$

Thus, from Eq. (3.95), the expression for its frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}, \tag{3.96}$$

which is a rational function in $e^{j\omega}$.

$$y_n \overset{LTI}{=\!=} \sum_k h_k\, x_{n-k} = \sum_k h_k\, e^{jw_0(n-k)}$$

$$= \left(\sum_k h_k\, e^{-jw_0 k} \right) \cdot e^{jw_0 n}$$

$$\overset{\triangle}{=} H(e^{jw_0})$$

fct of w_0 if you will

$$H(e^{jw}) \overset{\triangle}{=} \sum_n h_n\, e^{-jwn}$$ is the <u>frequency response</u> of the LTI with unit pulse resp. h_n, <u>IF IT EXISTS!</u>

$|H(e^{jw_0})| e^{j\angle H(e^{jw_0})}$

<u>magnitude</u> & <u>phase</u> of freq. resp.

Note: $\left| \sum_n h_n\, e^{-jwn} \right| \leq \sum_n \left| h_n\, e^{-jwn} \right| = \sum_n |h_n| \cdot \underbrace{|e^{-jwn}|}_{=1} = \sum_n |h_n|$

$$\Rightarrow \quad BIBO \iff \sum_n |h_n| < \infty \implies \begin{cases} H(e^{jw}) \text{ exists} \\ \text{freq. resp. exists} \end{cases}$$

The Concept of Filtering

- One application of an LTI discrete-time system is to **pass** certain frequency components in an input sequence without any distortion (if possible) and to **block** other frequency components

- Such systems are called digital filters and one of the main subjects of discussion in this course

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The Concept of Filtering

- The key to the filtering process is

$$x[n] = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences

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The Concept of Filtering

- Thus, by appropriately choosing the values of the magnitude function $\left|H(e^{j\omega})\right|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others

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The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$\left| H(e^{j\omega}) \right| \cong \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

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The Concept of Filtering

- We apply an input

$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n, \qquad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system

- Because of linearity, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

$$+ B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

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The Concept of Filtering

- As

$$\left| H(e^{j\omega_1}) \right| \cong 1, \quad \left| H(e^{j\omega_2}) \right| \cong 0$$

  the output reduces to

$$y[n] \cong A \left| H(e^{j\omega_1}) \right| \cos(\omega_1 n + \theta(\omega_1))$$

- Thus, the system acts like a **lowpass filter**

- In the following example, we consider the design of a very simple digital filter

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3.8.7 The Concept of Filtering

One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components. Such systems are called *digital filters* and are one of the main subjects of discussion in this text. The key to the filtering process is the inverse discrete-time Fourier transform given in Eq. (3.16) which expresses an arbitrary input sequence as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences. As a result, by appropriately choosing the values of magnitude function of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these sinusoidal sequences can be selectively heavily attenuated or filtered with respect to the others.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega,$$

We now explain the concept of filtering and then define the most commonly desired filter characteristics. To understand the mechanism behind the design of such a system, consider a real coefficient LTI discrete-time system characterized by a magnitude function

$$|H(e^{j\omega})| \cong \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c, \\ 0, & \omega_c < |\omega| < \pi. \end{cases} \qquad (3.106)$$

We apply an input $x[n] = A \cos \omega_1 n + B \cos \omega_2 n$ to this system, where $0 < \omega_1 < \omega_c < \omega_2 < \pi$. Because of linearity, it follows from Eq. (3.102) that the output $y[n]$ of this system is of the form

$$y[n] = A|H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1)) + B|H(e^{j\omega_2})| \cos(\omega_2 n + \theta(\omega_2)). \qquad (3.107)$$

Making use of Eq. (3.106) in Eq. (3.107), we get

$$y[n] \cong A|H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1)),$$

indicating the LTI discrete-time system acts like a lowpass filter.

Finite-Length
Discrete Transforms

Often, in practice, it is convenient to map a finite-length sequence from the time domain into a finite-length sequence of the same length in a different domain, and vice-versa. Such transformations are usually collectively called *finite-length transforms* and are the subject of this chapter. In the forward transform, the samples of the transform are unique and represented as a linear combination of the samples of the time-domain sequence. The original time-domain sequence can be obtained by applying an inverse transform in which the time-domain samples are expressed as a linear combination of the samples of its transform-domain representation.

In some applications, a very long length time-domain sequence is broken up into a set of short-length time-domain sequences and a finite-length transform is applied to each short-length sequence. The transformed sequences are next processed in the transform domain, and their time-domain equivalents are generated by applying the inverse transforms. The processed short-length sequences are then grouped together appropriately to develop the final long-length sequence.

A variety of finite-dimensional transforms have been advanced, and a discussion on each of these transforms is beyond the scope of this book. We restrict our attention in this chapter to the so-called class of *orthogonal transforms*. In particular, we discuss here three such transforms, namely, the *discrete Fourier transform*, the *discrete cosine transform*, and the *Haar transform*. The former transform is widely used in a number of digital signal processing applications, whereas the latter two find applications primarily in signal compression.

5.1 Orthogonal Transforms

Let $x[n]$ denote a length-N time-domain sequence with $\mathcal{X}[k]$ denoting the coefficients of its N-point orthogonal transform. Then, a general form of the orthogonal transform pair is of the form

$$\mathcal{X}[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k, n], \qquad 0 \leq k \leq N - 1, \tag{5.1}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{X}[k]\psi[k, n], \qquad 0 \leq n \leq N - 1. \tag{5.2}$$

Equation (5.1) usually is referred to as the *analysis equation*, whereas Eq. (5.2) is referred to as the *synthesis equation*. In these equations, $\psi[k, n]$, called the *basis sequences*, are also length-N sequences in both domains. In the class of finite-dimensional transforms we shall deal with in this chapter, the basis sequences satisfy the condition

$$\frac{1}{N} \sum_{n=0}^{N-1} \psi[k, n]\psi^*[\ell, n] = \begin{cases} 1, & \ell = k, \\ 0, & \ell \neq k. \end{cases} \tag{5.3}$$

5.2   The Discrete Fourier Transform

In this section, we define the discrete Fourier transform, usually known as the DFT, and develop the inverse transformation, often abbreviated as IDFT. We relate the DFT to the Fourier transform of the time-domain sequence, review its major properties, and study especially two of its unique properties in the following section. Several important applications of the DFT, such as the numerical computation of the Fourier transform and implementation of linear convolution, are discussed in a later section.

5.2.1   Definition

The *discrete Fourier transform* (DFT) of the length-N time-domain sequence $x[n]$ is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \qquad 0 \le k \le N - 1, \qquad (5.7)$$

obtained by setting the basis sequences in Eq. (5.1) as

$$\psi[k, n] = e^{j2\pi kn/N}, \qquad (5.8)$$

which are complex exponential sequences. As a result, the DFT coefficients, $X[k]$, in general are complex numbers even when $x[n]$ are real. It can be easily shown that the basis sequences $e^{j2\pi kn/N}$ are orthogonal:

that is, they satisfy the condition of Eq. (5.3). To this end, we observe that

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi \ell n/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N}.$$

By setting $u = e^{j2\pi(k-\ell)/N}$, the right-hand side of the above equation reduces to a finite sum of the form

$$S_{N-1} = \sum_{n=0}^{N-1} u^n.$$

It follows from the above that

$$\sum_{n=0}^{N} u^n = 1 + u \sum_{n=0}^{N-1} u^n = 1 + u S_{N-1} = S_{N-1} + u^N,$$

which, when solved, yields

$$S_{N-1} = \frac{1 - u^N}{1 - u}. \qquad (5.9)$$

Substituting $u = e^{j2\pi(k-\ell)/N}$ in the above equation we get

$$\sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N} = \frac{1 - e^{j2\pi(k-\ell)}}{1 - e^{j2\pi(k-\ell)/N}}. \qquad (5.10)$$

For $k \neq \ell$, the numerator of the right-hand side of the above equation is equal to zero. For $k = \ell + rN$, the right-hand side of the above equation is of the form $0/0$. However, it can be seen from the left-hand side that for $k = \ell$, the sum is equal to N. Hence,

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N} = \begin{cases} 1, & \text{for } k = \ell + \\ 0, & \text{for } k \neq \ell, \end{cases}$$

verifying the orthogonality property of the basis sequences $e^{j2\pi kn/N}$. It follows from Eq. (5.7) that the

verifying the orthogonality property of the basis sequences $e^{j2\pi kn/N}$. It follows from Eq. (5.7) that the DFT $X[k]$ is a length-N sequence in the transform domain. Often, the length-N DFT sequence is referred to as the *N-point DFT*. Applying the commonly used notation

$$W_N = e^{-j2\pi/N},$$
(5.12)

we can rewrite Eq. (5.7) as

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \qquad 0 \le k \le N - 1.$$
(5.13)

The *inverse discrete Fourier transform* (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le N - 1,$$
(5.14)

obtained using the basis sequences of Eq. (5.8) in Eq. (5.2). As can be seen from the above expression, the inverse DFT $x[n]$ can be a complex sequence even when the DFT $X[k]$ is a real sequence.

Equations (5.13) and (5.14) constitute a discrete Fourier transform pair for the sequence $x[n]$. A discrete Fourier transform pair will often be denoted as

$$x[n] \xleftrightarrow{\text{DFT}} X[k].$$
(5.15)

The DFT computation is illustrated in Examples (5.1) and (5.2).

Verifying : Examples 5.1 and 5.2

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \leq n \leq N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \leq k \leq N-1$$

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Discrete Fourier Transform

- <u>Example</u> - Consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \leq n \leq m-1, \; m+1 \leq n \leq N-1 \end{cases}$$

- Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$$0 \leq k \leq N-1$$

5.2.2 Matrix Relations

The DFT samples defined in Eq. (5.13) can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}, \tag{5.25}$$

where $\mathbf{X}$ is the vector composed of the N DFT samples,

$$\mathbf{X} = [\, X[0] \quad X[1] \quad \cdots \quad X[N-1]\,]^T, \tag{5.26}$$

$\mathbf{x}$ is the vector of N input samples,

$$\mathbf{x} = [\, x[0] \quad x[1] \quad \cdots \quad x[N-1]\,]^T, \tag{5.27}$$

and $\mathbf{D}_N$ is the $N \times N$ *DFT matrix* given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}. \tag{5.28}$$

Likewise, the IDFT relations can be expressed in matrix form as

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \mathbf{D}_N^{-1} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}, \tag{5.29}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ IDFT matrix given by

$$\mathbf{D}_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}. \tag{5.30}$$

It follows from Eqs. (5.28) and (5.30) that

$$\mathbf{D}_N^{-1} = \frac{1}{N} \mathbf{D}_N^*. \tag{5.31}$$

5.3 Relation Between the Fourier Transform and the DFT and Their Inverses

We now examine the explicit relation between the Fourier transform and the N-point DFT of a length-N sequence and the relation between the Fourier transform of a length-M sequence and the N-point DFT obtained by sampling the Fourier transform.

5.3.1 Relation with Discrete-Time Fourier Transform

From Eq. (3.12), the Fourier transform $X(e^{j\omega})$ of the length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}. \tag{5.34}$$

By uniformly sampling $X(e^{j\omega})$ at N equally spaced frequencies $\omega_k = 2\pi k/N, 0 \leq k \leq N-1$, on the ω-axis between $0 \leq \omega < 2\pi$, we get from the above equation

$$X(e^{j\omega})\big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi k/N}, \qquad 0 \leq k \leq N-1. \tag{5.35}$$

Comparing Eq. (5.35) with Eq. (5.7), we observe that the N-point DFT sequence $X[k]$ is precisely the set of frequency samples of the Fourier transform $X(e^{j\omega})$ of the length-N sequence $x[n]$ at N equally spaced frequencies $\omega_k = 2\pi k/N, 0 \leq k \leq N-1$. Hence, Eq. (5.7) represents a frequency-domain representation of the sequence $x[n]$.[1] Since the computation of the DFT samples involve a finite sum, for time-domain sequences with finite sample values, the DFT always exists.

Because of the explicit relation between the DFT samples and the frequency samples of the Fourier transform, the normalized angular frequency associated with the index k of the DFT sample $X[k]$ is $2\pi k/N$ radians. For example, for $N = 32$, the sample index 11 represents the normalized angular frequency $\omega = 11\pi/16$.

5.3.2 Numerical Computation of the Fourier Transform Using the DFT

The DFT provides a practical approach to the numerical computation of the Fourier transform of a finite-length sequence, particularly if fast algorithms are available for the computation of the DFT. Let $X(e^{j\omega})$ be the Fourier transform of a length-N sequence $x[n]$. We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k/M, 0 \le k \le M-1$, where $M \gg N$:

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}. \tag{5.36}$$

Define a new sequence $x_e[n]$ obtained from $x[n]$ by augmenting with $M - N$ zero-valued samples:

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le M-1. \end{cases} \tag{5.37}$$

Making use of $x_e[n]$ in Eq. (5.36), we arrive at

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n]e^{-j2\pi kn/M}, \tag{5.38}$$

which is seen to be an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$. The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2.

The MATLAB function freqz, described in Section 3.6, employs the above approach to evaluate the frequency response of a rational Fourier transform expressed as a rational function in $e^{-j\omega}$ at a prescribed set of discrete frequencies. It computes the DFTs of the numerator and the denominator separately by considering each as finite-length sequences and then expresses the ratio of the DFT samples at each frequency point to evaluate the Fourier transform.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le N-1. \qquad\qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad 0 \le n \le N-1,$$

As can be seen from Eqs. (5.13) and (5.14), the computation of the DFT and the IDFT requires, respectively, approximately N^2 complex multiplications and $N(N-1)$ complex additions. However, elegant methods have been developed to reduce the computational complexity to about $N(\log_2 N)$ operations. These techniques are usually called fast Fourier transform (FFT) algorithms and are discussed in Sections 11.3.2 and 11.4. As a result of the availability of these fast algorithms, the DFT and the IDFT, and their variations, are often used in digital signal processing applications for various purposes.

5.2.3   DFT Computation Using MATLAB

There are four built-in functions in MATLAB for the computation of the DFT and the IDFT:

```
fft(x),          fft(x,M),          ifft(X),          ifft(X,M)
```

All of these functions make use of FFT algorithms, which are computationally highly efficient compared to the direct computation of DFT and the inverse DFT. In addition, the function `dftmtx(N)` in the *Signal Processing Toolbox* of MATLAB can be used to compute the $N \times N$ DFT matrix $\mathbf{D}_N$, defined in Eq. (5.28). To compute the inverse of the $N \times N$ DFT matrix, one can use the function `conj(dftmtx(N))/N`.
    We illustrate the application of the above M-files in Examples 5.3 and 5.4.

    The M-file `fftshift` shifts the zero-frequency sample at the frequency index $k = 0$ to the center of the spectrum and is often useful in visualizing the spectrum of a sequence.

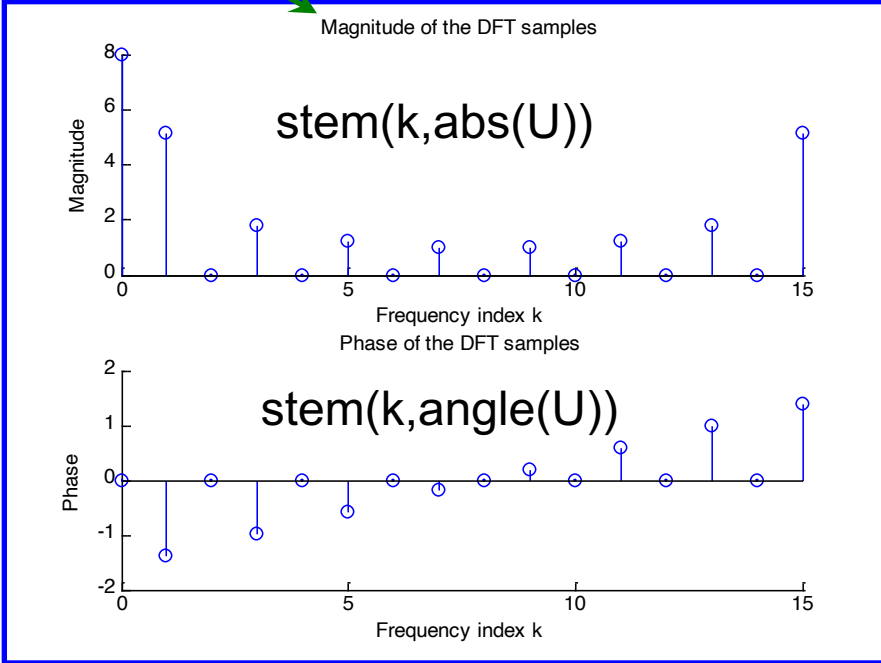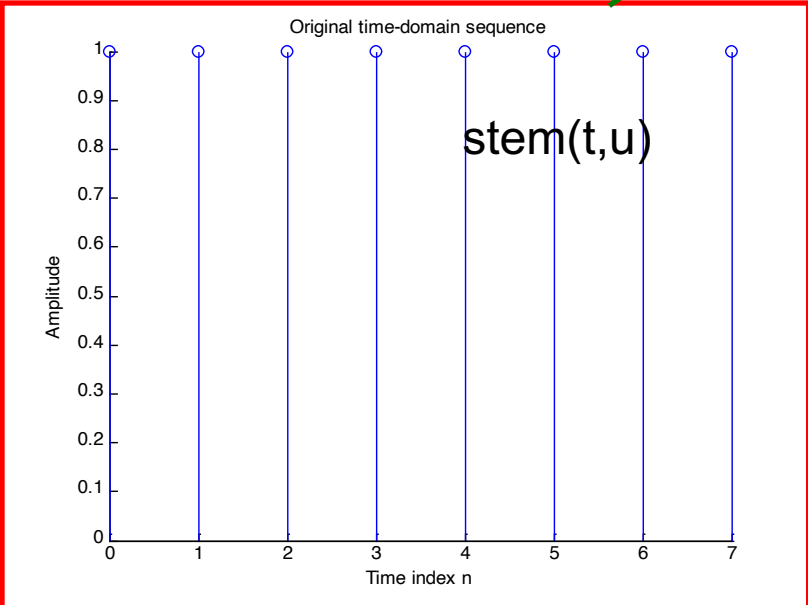Verifying : Examples 5.3 and 5.4

DFT Computation Using MATLAB

- The functions to compute the DFT and the IDFT are `fft` and `ifft`

- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation

- Programs 5_1.m and 5_2.m illustrate the use of these functions

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The DFT provides a practical approach to the numerical computation of the Fourier transform of a finite-length sequence, particularly if fast algorithms are available for the computation of the DFT. Let $X(e^{j\omega})$ be the Fourier transform of a length-N sequence $x[n]$. We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k/M$, $0 \leq k \leq M-1$, where $M \gg N$:

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}.$$

U = fft(u,M)

M=16, the length of the DFT

Original time-domain sequence

stem(t,u)

Magnitude of the DFT samples

stem(k,abs(U))

Phase of the DFT samples

stem(k,angle(U))

u = [ones(1,N)];

N=8, the length of the sequence

Read in the length K=8 of the DFT and the desired
% length N =13 of the IDFT

Original DFT samples

$$V[k] = \begin{cases} k/K, & 0 \le k \le K - 1, \\ 0, & \text{otherwise.} \end{cases}$$

v = ifft(V,N)

(a)

Real part of the time-domain samples

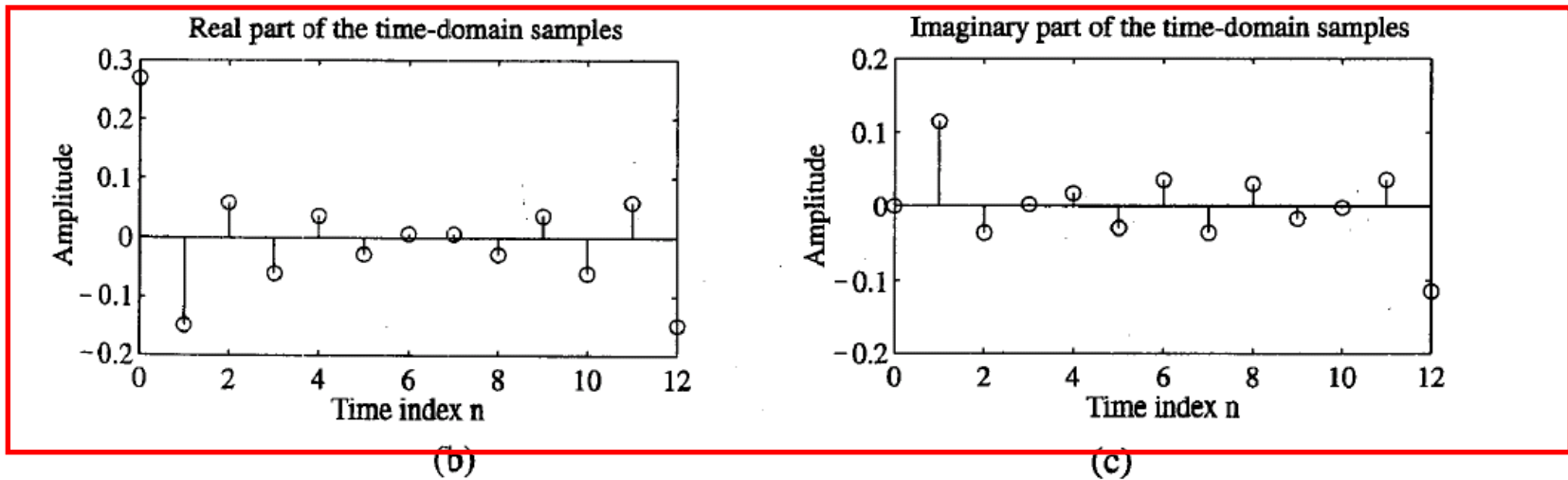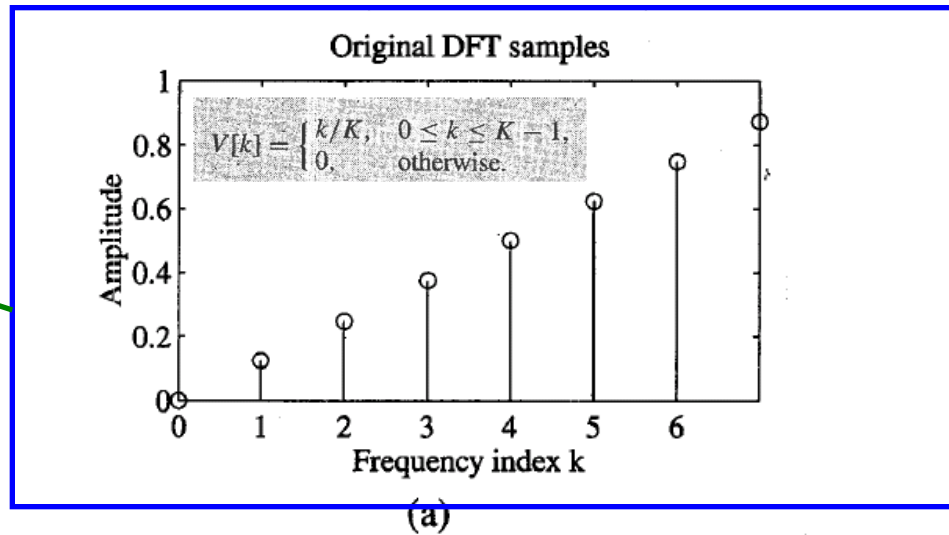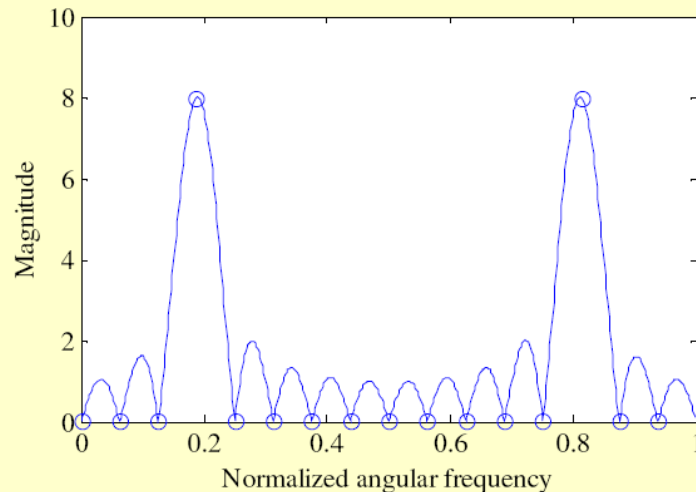Imaginary part of the time-domain samples

(b)

(c)

Figure 5.2: (a) Original DFT sequence of length $K = 8$ and (b) its 13-point IDFT.

DFT Computation Using MATLAB

- <u>Example</u> - Program 5_3.m can be used to compute the DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

as shown below

○ indicates DFT samples

```
% Program 5_3
% Numerical Computation of Fourier transform Using DFT

k = 0:15;
w = 0:511;
x = cos(2*pi*k*3/16);% Generate the length-16 sinusoidal sequence

X = fft(x); % Compute its 16-point DFT=fft(x,16)

XE = fft(x,512); % Compute its 512-point DFT

% Plot the frequency response and the 16-point DFT samples

plot(k/16,abs(X),'o', w/512,abs(XE))

xlabel('\omega/\pi');
ylabel('Magnitude')
```