EENG 479 : Digital Signal Processing (DSP)

Lecture #13: FIR filters (Fixed Window)

Prof. Mohab A. Mangoud
Professor of Wireless Communications (Networks, IoT and AI)
University of Bahrain, College of Engineering
Department of Electrical and Electronics Engineering
P.O.Box 32038- Kingdom of Bahrain
mmangoud@uob.edu.bh
http://mangoud.com

Basic Approaches to FIR Digital Filter Design

- Unlike IIR digital filter design, FIR filter design does not have any connection with the design of analog filters.

- The design of FIR filters is therefore based on a direct approximation of the specified magnitude response, with linear phase response

- recall a causal FIR transfer function *H(z)* of length N+1 is a polynomial in z^{-1} of degree N and the corresponding frequency response is given by:

$$H(e^{j\omega}) = \sum_{n=0}^{N} h[n]e^{-j\omega n}.$$

the design of FIR filter of length N+1 can be accomplished by finding either the impulse response sequence {h[n]} or N+1 samples of its frequency response $H(e^{j\omega t})$

Least Integral-Squared Error Design of FIR Filters

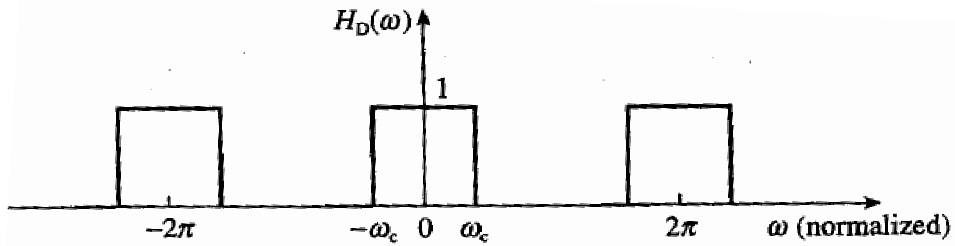
- Let $H_d(e^{j\omega})$ denote the desired frequency response
- Since $H_d(e^{j\omega})$ is a periodic function of ω with a period 2π, it can be expressed as a Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

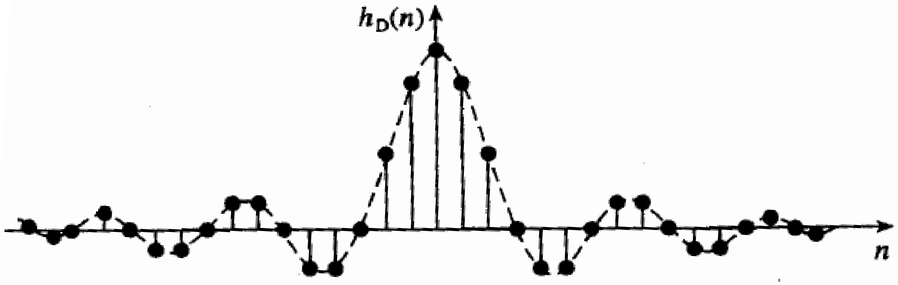
where

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \le n \le \infty$$

- In general, $H_d(e^{j\omega})$ is piecewise constant with sharp transitions between bands
- In which case, $\{h_d[n]\}$ is of infinite length and noncausal
- Objective - Find a finite-duration $\{h_t[n]\}$ of length $2M+1$ whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$ in some sense



(a)



(b)

(a) Ideal frequency response of a lowpass filter. (b) Impulse response of the ideal lowpass filter.

$$h_D(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \times e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{2f_c \sin(n\omega_c)}{n\omega_c}, \quad n \neq 0, -\infty \leqslant n \leqslant \infty$$

$$= 2f_c, \quad n = 0 \text{ (using L'Hôpital's rule)}$$

Not BIBO, Not Causal : we need to shift it and chop off tails (ie : keep the most significan parts. One most commonly used approximation criterion is to :**<u>minimize the integral-squared error.</u>**

$$\Phi_R = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_t(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 d\omega,$$

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n}.$$

- Using Parseval's relation we can write

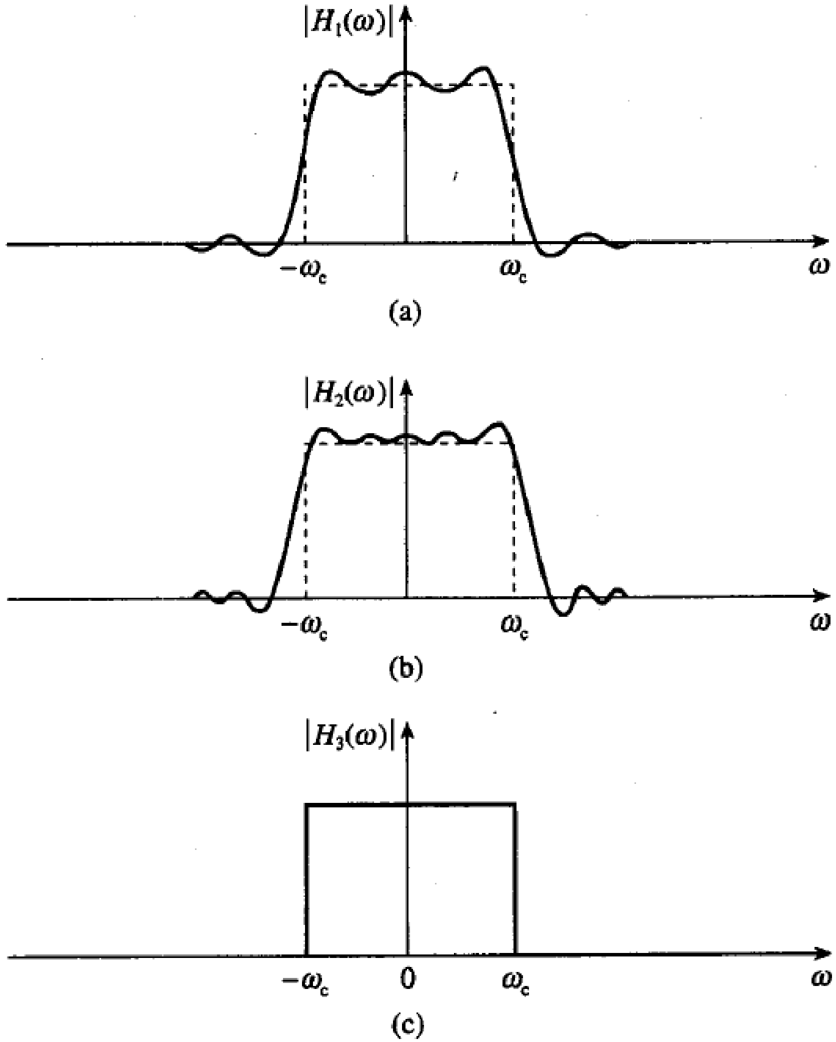
$$\Phi = \sum_{n=-\infty}^{\infty} |h_t[n] - h_d[n]|^2$$

$$= \sum_{n=-M}^{M} |h_t[n] - h_d[n]|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

- It follows from the above that Φ is minimum when $h_t[n] = h_d[n]$ for $-M \le n \le M$
- $\Rightarrow$ Best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncation
- A causal FIR filter with an impulse response $h[n]$ can be derived from $h_t[n]$ by delaying:

$$h[n] = h_t[n - M]$$

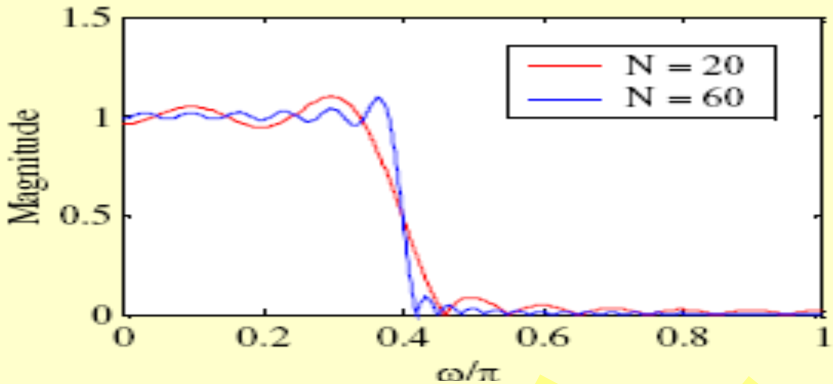
- The causal FIR filter $h[n]$ has the same magnitude response as $h_t[n]$ and its phase response has a linear phase shift of ωM radians with respect to that of $h_t[n]$



5 Effects on the frequency response of truncating the ideal impulse response to (a) 13 coefficients, (b) 25 coefficients and (c) an infinite number of coefficients.

Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths
- Height of the largest ripples remain the same independent of length
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters

- Gibbs phenomenon can be explained by treating the truncation operation as an windowing operation:

$$h_t[n] = h_d[n] \cdot w[n]$$

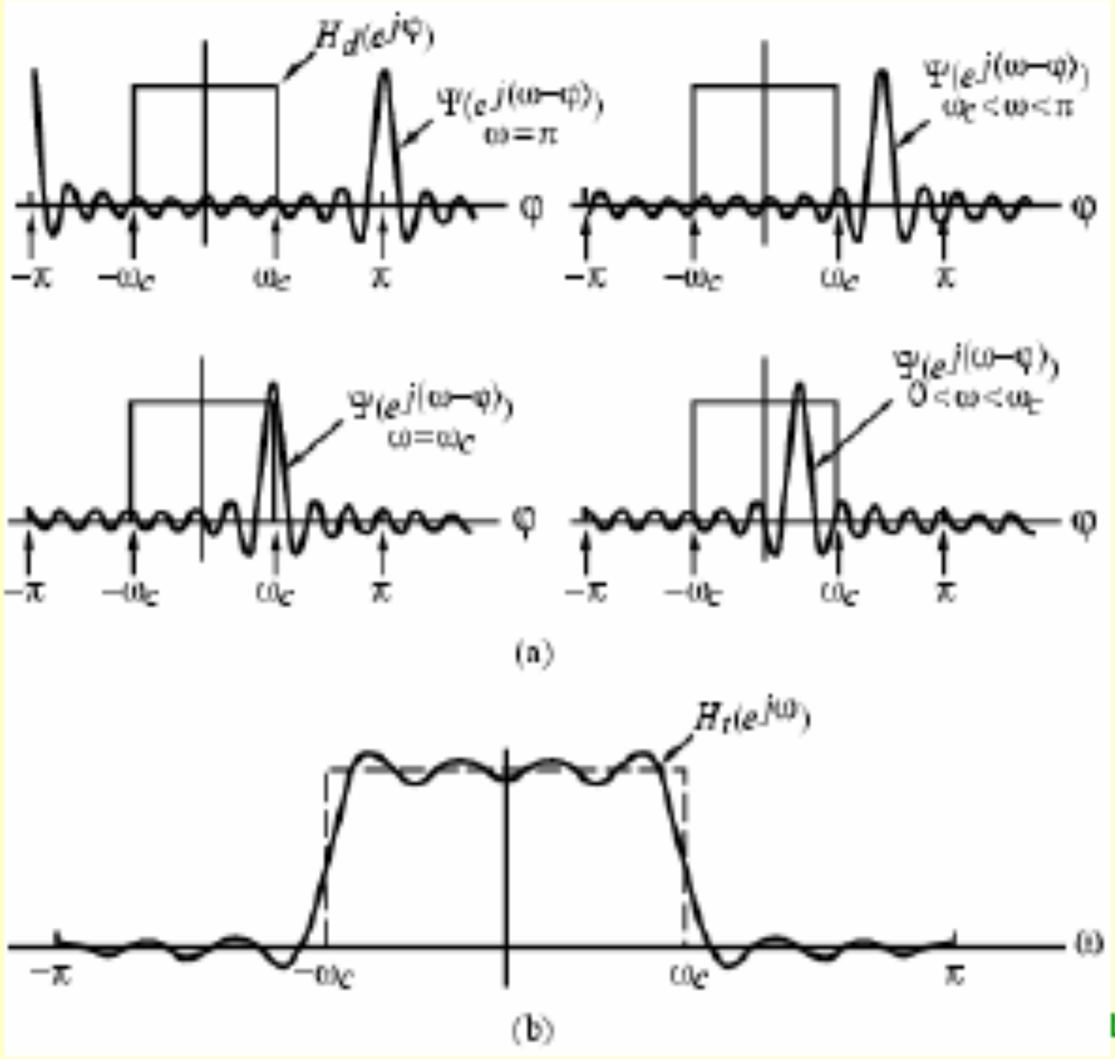
- In the frequency domain

Convolution

$$H_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\varphi}) \Psi(e^{j(\omega-\varphi)}) d\varphi$$

- where $H_t(e^{j\omega})$ and $\Psi(e^{j\omega})$ are the DTFTs of $h_t[n]$ and $w[n]$, respectively

- Thus $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of $H_d(e^{j\omega})$ with $\Psi(e^{j\omega})$



- If $\Psi(e^{j\omega})$ is a very narrow pulse centered at $\omega = 0$ (ideally a delta function) compared to variations in $H_d(e^{j\omega})$, then $H_t(e^{j\omega})$ will approximate $H_d(e^{j\omega})$ very closely

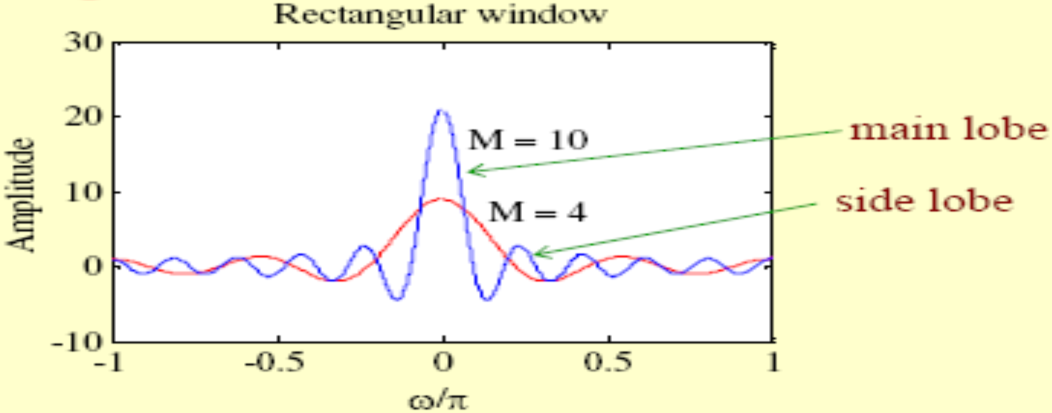
Large window

- Length $2M+1$ of $w[n]$ should be very large
- On the other hand, length $2M+1$ of $h_t[n]$ should be as small as possible to reduce computational complexity
- A **rectangular window** is used to achieve simple truncation:

$$w_R[n] = \begin{cases} 1, & 0 \le |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

- Presence of oscillatory behavior in $H_t(e^{j\omega})$ is basically due to:
  - 1) $h_d[n]$ is infinitely long and not absolutely summable, and hence filter is unstable
  - 2) Rectangular window has an abrupt transition to zero

- Oscillatory behavior can be explained by examining the DTFT $\Psi_R(e^{j\omega})$ of $w_R[n]$:



- $\Psi_R(e^{j\omega})$ has a **main lobe** centered at $\omega = 0$
- Other ripples are called **sidelobes**

- Main lobe of $\Psi_R(e^{j\omega})$ characterized by its width $4\pi/(2M+1)$ defined by first zero crossings on both sides of $\omega = 0$
- As M increases, width of main lobe decreases as desired
- Area under each lobe remains constant while width of each lobe decreases with an increase in M
- Ripples in $H_t(e^{j\omega})$ around the point of discontinuity occur more closely but with no decrease in amplitude as M increases

- Rectangular window has an abrupt transition to zero outside the range $-M \leq n \leq M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

(1) Using a window that tapers smoothly to zero at each end, or

(2) Providing a smooth transition from passband to stopband in the magnitude specifications

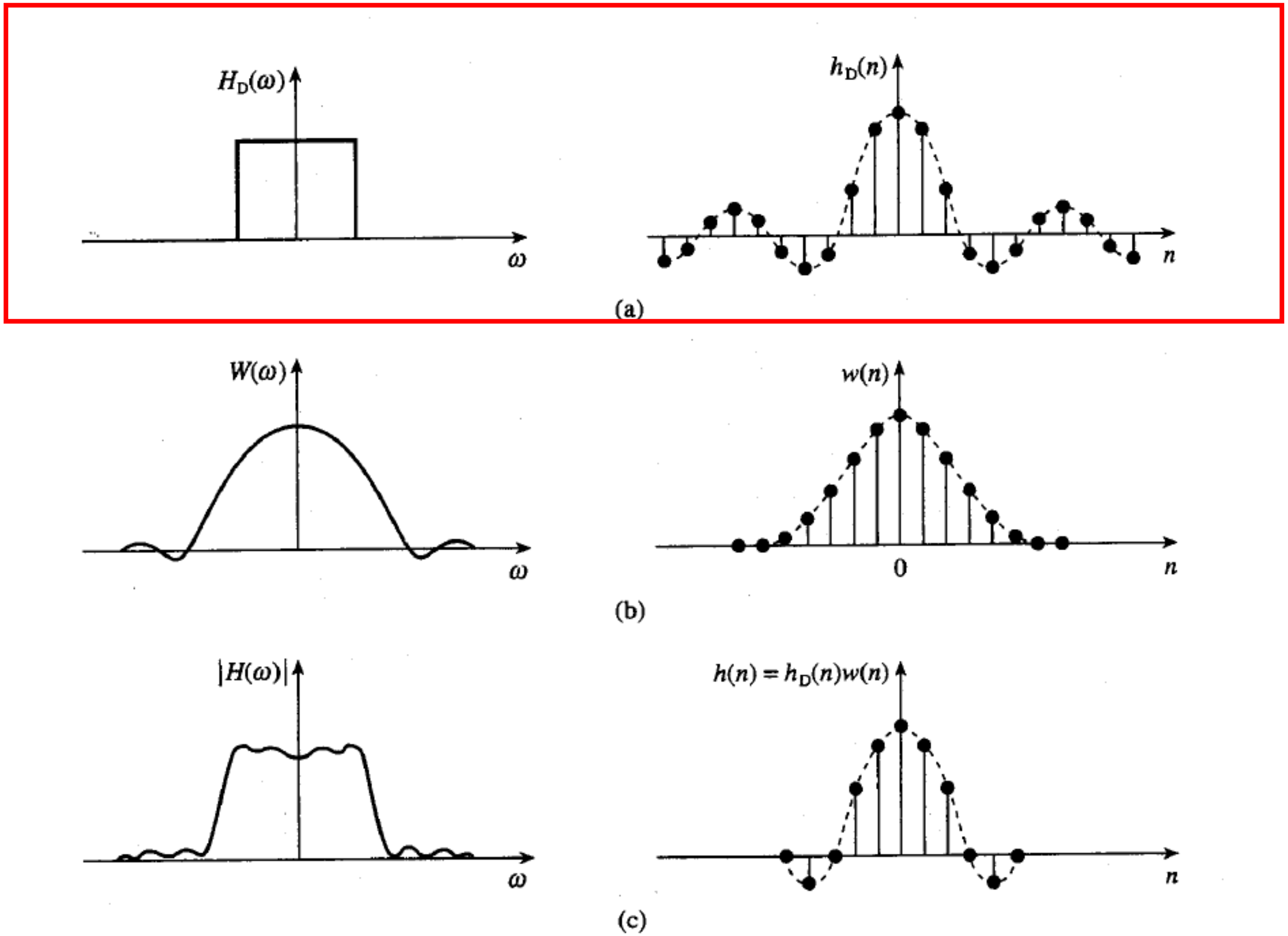
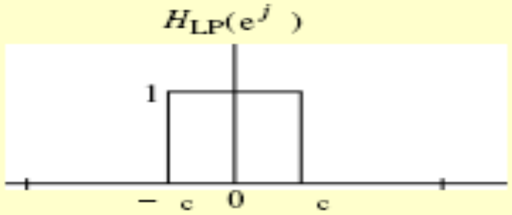


Figure 7.6 An illustration of how the filter coefficients, $h(n)$, are determined by the window method.

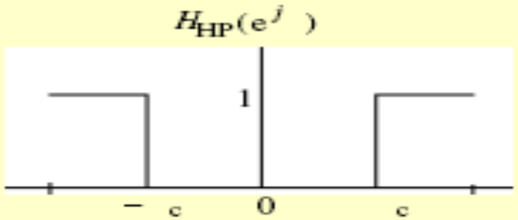
Impulse Responses of Ideal Filters

- Ideal lowpass filter -



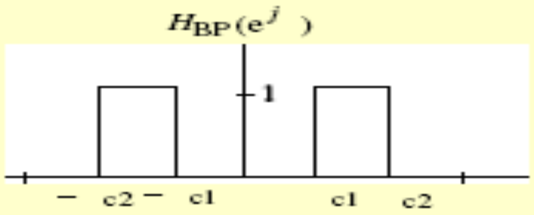
$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$

- Ideal highpass filter -



$$h_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

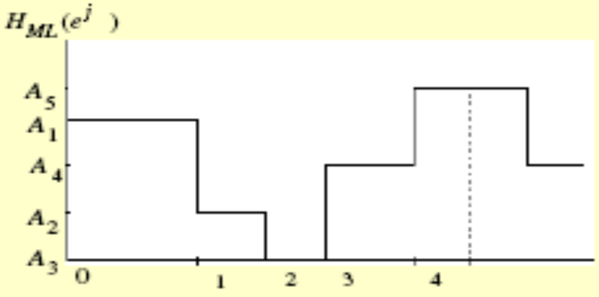
- Ideal bandpass filter -



$$h_{BP}[n] = \begin{cases} \frac{\sin(\omega_{c2} n)}{\pi n} - \frac{\sin(\omega_{c1} n)}{\pi n}, & n \neq 0 \\ \frac{\omega_{c2}}{\pi} - \frac{\omega_{c1}}{\pi}, & n = 0 \end{cases}$$

Impulse Responses of Ideal Filters

- Ideal multiband filter -



$$H_{ML}(e^{j\omega}) = A_k,$$
$$\omega_{k-1} \le \omega \le \omega_k,$$
$$k = 1, 2, \ldots, L$$

$$h_{ML}[n] = \sum_{\ell=1}^{L} (A_\ell - A_{\ell+1}) \cdot \frac{\sin(\omega_L n)}{\pi n}$$

- Ideal discrete-time Hilbert transformer -

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0 \\ -j, & 0 < \omega < \pi \end{cases}$$

$$h_{HT}[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 2/\pi n, & \text{for } n \text{ odd} \end{cases}$$

- Ideal discrete-time differentiator -

$$H_{DIF}(e^{j\omega}) = j\omega, \quad 0 \le |\omega| \le \pi$$

$$h_{DIF}[n] = \begin{cases} 0, & n = 0 \\ \dfrac{\cos \pi n}{n}, & n \ne 0 \end{cases}$$

Table 7.2 Summary of ideal impulse responses for standard frequency selective filters.

	Ideal impulse response, $h_D(n)$	
Filter type	**$h_D(n), n \neq 0$**	**$h_D(0)$**
Lowpass	$2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$2f_c$
Highpass	$-2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$1-2f_c$
Bandpass	$2f_2 \frac{\sin(n\omega_2)}{n\omega_2} - 2f_1 \frac{\sin(n\omega_1)}{n\omega_1}$	$2(f_2-f_1)$
Bandstop	$2f_1 \frac{\sin(n\omega_1)}{n\omega_1} - 2f_2 \frac{\sin(n\omega_2)}{n\omega_2}$	$1-2(f_2-f_1)$

f_c, f_1 and f_2 are the normalized passband or stopband edge frequencies; N is the length of filter.

FIR Filter Design Based on Windowed Fourier Series

Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity
- Hann:

$$w[n] = 0.5 + 0.5\cos(\frac{2\pi n}{2M+1}), \quad -M \le n \le M$$

- Hamming:

$$w[n] = 0.54 + 0.46\cos(\frac{2\pi n}{2M+1}), \quad -M \le n \le M$$

- Blackman:

$$w[n] = 0.42 + 0.5\cos(\frac{2\pi n}{2M+1}) + 0.08\cos(\frac{4\pi n}{2M+1})$$

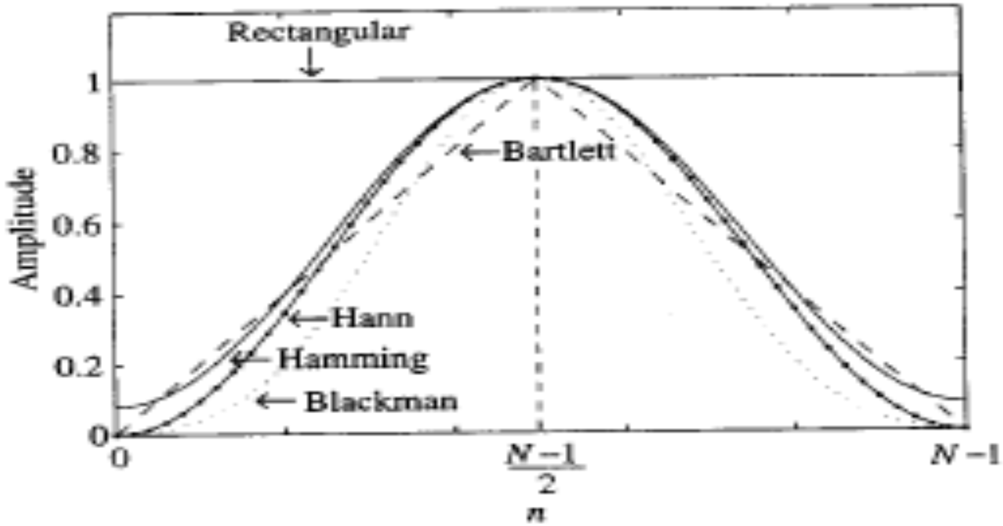


Figure 10.6: Plots of the fixed windows shown with solid lines for clarity.

Generated using Matlab hann, hamming, blackman, bartlett commands (plotted as continues for clarity)

windows of length $N = 2M + 1$ listed below [Sar93]:[2]

$$\textit{Bartlett}:\ w[n] = 1 - \frac{|n|}{M+1}, \qquad -M \le n \le M, \tag{10.29}$$

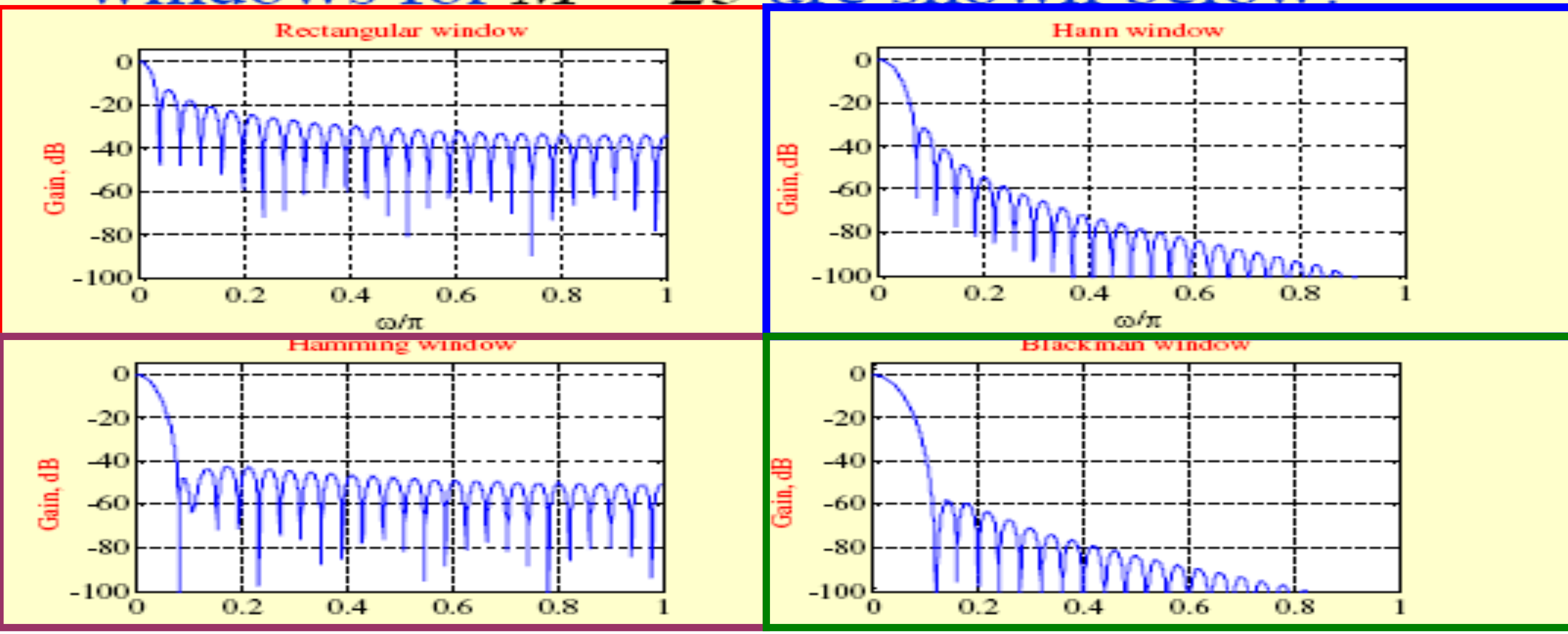
$$\textit{Hann}:\ w[n] = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n}{2M+1}\right)\right], \qquad -M \le n \le M, \tag{10.30}$$

$$\textit{Hamming}:\ w[n] = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right), \qquad -M \le n \le M, \tag{10.31}$$

$$\textit{Blackman}:\ w[n] = 0.42 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right)] + 0.08\cos\left(\frac{4\pi n}{2M+1}\right), \qquad -M \le n \le M. \tag{10.32}$$

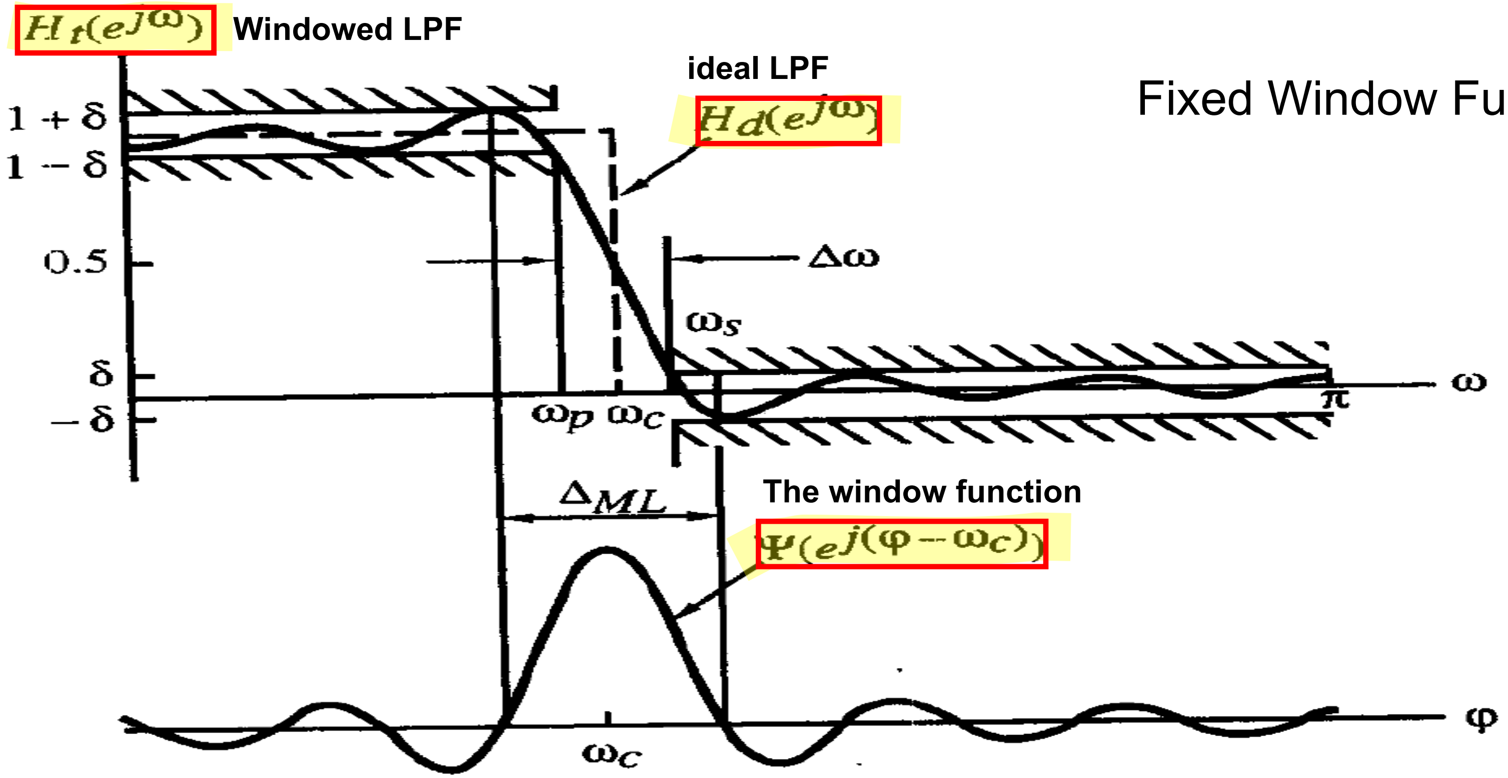
Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:



- Magnitude spectrum of each window characterized by a main lobe centered at $\omega = 0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:
- **Main lobe width**
- **Relative sidelobe level**
- **Main lobe width** Δ_{ML} - given by the distance between zero crossings on both sides of main lobe
- **Relative sidelobe level** $A_{s\ell}$ - given by the difference in dB between amplitudes of largest sidelobe and main lobe

Fixed Window Function



- Observe $H_t(e^{j(\omega_c+\Delta\omega)}) + H_t(e^{j(\omega_c-\Delta\omega)}) \cong 1$
- Thus, $H_t(e^{j\omega_c}) \cong 0.5$
- Passband and stopband ripples are the same
- Distance between the locations of the maximum passband deviation and minimum stopband value $\cong \Delta_{ML}$

- Width of transition band

$$\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$$

- To ensure a fast transition from passband to stopband, window should have a very small main lobe width
- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small
- Unfortunately, these two requirements are contradictory

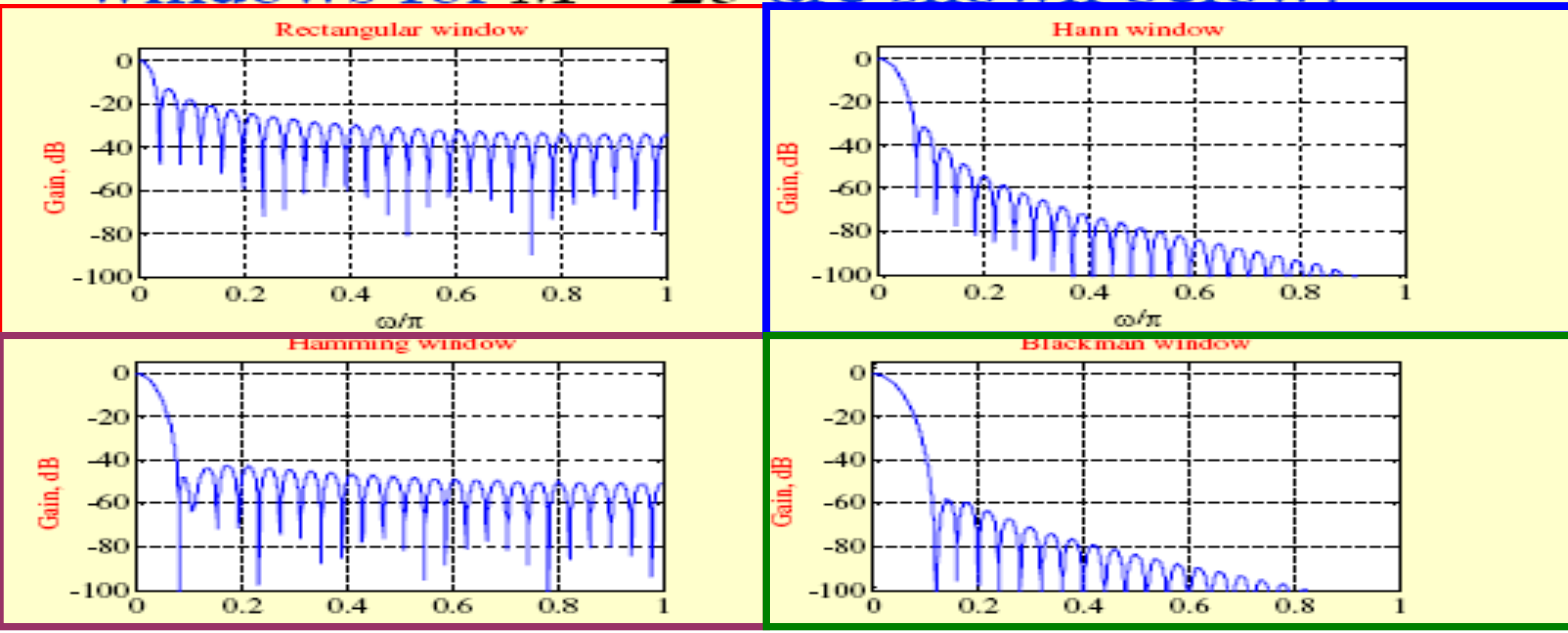
- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant
- In addition,

$$\Delta\omega \approx \frac{c}{M}$$

where c is a constant for most practical purposes

Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:



- **Rectangular window** - $\Delta_{ML} = 4\pi/(2M+1)$
  $A_{s\ell} = 13.3$ dB, $\alpha_s = 20.9$ dB, $\Delta\omega = 0.92\pi/M$
- **Hann window** - $\Delta_{ML} = 8\pi/(2M+1)$
  $A_{s\ell} = 31.5$ dB, $\alpha_s = 43.9$ dB, $\Delta\omega = 3.11\pi/M$
- **Hamming window** - $\Delta_{ML} = 8\pi/(2M+1)$
  $A_{s\ell} = 42.7$ dB, $\alpha_s = 54.5$dB, $\Delta\omega = 3.32\pi/M$
- **Blackman window** - $\Delta_{ML} = 12\pi/(2M+1)$
  $A_{s\ell} = 58.1$ dB, $\alpha_s = 75.3$ dB, $\Delta\omega = 5.56\pi/M$

Table 10.2: Properties of some fixed window functions.[4]

Type of Window	Main Lobe Width Δ_{ML}	Relative Sidelobe Level $A_{s\ell}$	Minimum Stopband Attenuation	Transition Bandwidth $\Delta\omega$
Rectangular	$4\pi/(2M+1)$	13.3 dB	20.9 dB	$0.92\pi/M$
Barlett	$4\pi/(M+1)$	26.5 dB	See text	See text
Hann	$8\pi/(2M+1)$	31.5 dB	43.9 dB	$3.11\pi/M$
Hamming	$8\pi/(2M+1)$	42.7 dB	54.5 dB	$3.32\pi/M$
Blackman	$12\pi/(2M+1)$	58.1 dB	75.3 dB	$5.56\pi/M$

Table 10.2 summarizes the essential properties of the above window functions, except the Bartlett window. For the latter window, the stopband edge is difficult to determine as the frequency response of the filter designed using this window has no unit circle zeros, and as a result, the value of the stopband attenuation and the expression for the transition bandwidth are not precisely known. The Bartlett window finds applications in spectral estimation.

- Filter Design Steps -

(1) Set

$$\omega_c = (\omega_p + \omega_s)/2$$

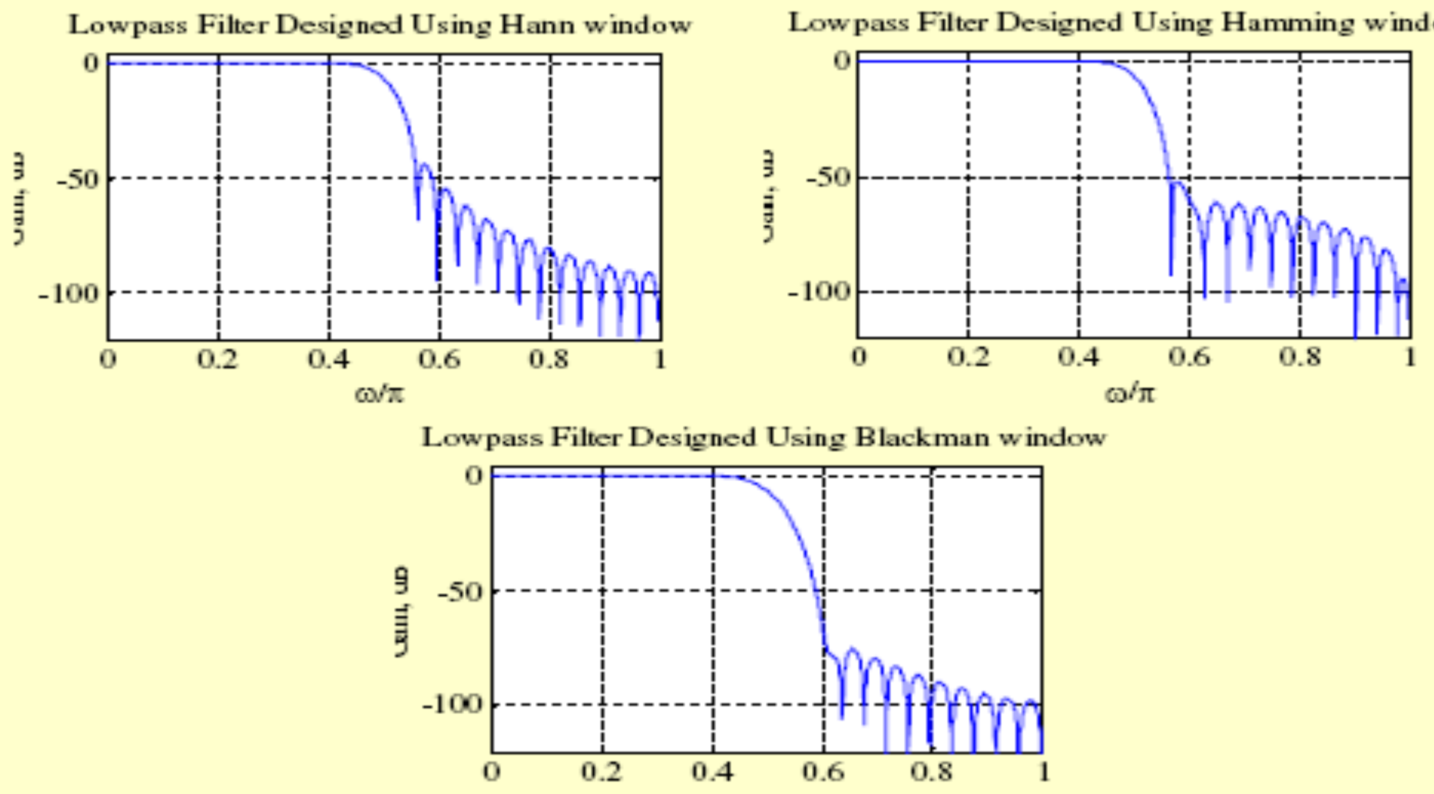
(2) Choose window based on specified α_s

(3) Estimate M using

$$\Delta\omega \approx \frac{c}{M}$$

FIR Filter Design Example

- Lowpass filter of length 51 and $\omega_c = \pi / 2$



- An increase in the main lobe width is associated with an increase in the width of the transition band
- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

EEENG 479 : Digital Signal Processing (DSP)

Lecture #14: FIR filters (Adjustable Window)

Prof. Mohab A. Mangoud
Professor of Wireless Communications (Networks, IoT and AI)
University of Bahrain, College of Engineering
Department of Electrical and Electronics Engineering
P.O.Box 32038- Kingdom of Bahrain
mmangoud@uob.edu.bh
http://mangoud.com

Adjustable Window Functions

- **Dolph-Chebyshev Window** -

$$w[n] = \frac{1}{2M+1}\left[\frac{1}{\gamma} + 2\sum_{k=1}^{M} T_k\left(\beta \cos\frac{k}{2M+1}\right)\cos\frac{2nk\pi}{2M+1}\right], \quad -M \le n \le M$$

where
$$\gamma = \frac{\text{amplitude of sidelobe}}{\text{main lobe amplitude}}$$

$$\beta = \cosh\left(\frac{1}{2M}\cosh^{-1}\frac{1}{\gamma}\right)$$

and
$$T_\ell(x) = \begin{cases} \cos(\ell \cos^{-1} x), & |x| \le 1 \\ \cosh(\ell \cosh^{-1} x), & |x| > 1 \end{cases}$$

Unlike ripples of the filter designed using one of the fixed windows functions which are fixed

- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately
- Filter order is estimated using

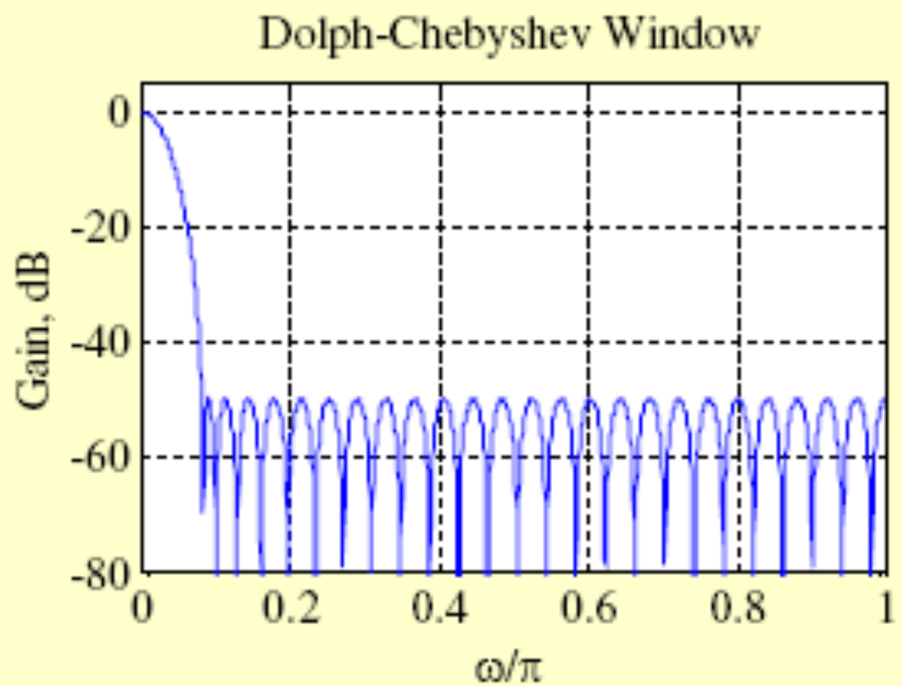
$$N = \frac{2.056\alpha_s - 16.4}{2.85(\Delta\omega)}$$

N=2M

where $\Delta\omega$ is the normalized transition bandwidth, e.g, for a lowpass filter

$$\Delta\omega = \omega_s - \omega_p$$

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below



Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height
- Stopband approximation error of filters designed have essentially equiripple behavior
- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

Adjustable Window Functions

- **Kaiser Window** -

The most commonly used

$$w[n] = \frac{I_0\{\beta\sqrt{1-(n/M)^2}\}}{I_0(\beta)}, \quad -M \le n \le M$$

where β is an adjustable parameter and $I_0(u)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty}\left[\frac{(u/2)^r}{r!}\right]^2$$

N=2M

- Note $I_0(u) > 0$ for $u > 0$
- In practice $I_0(u) \cong 1 + \sum_{r=1}^{20}\left[\frac{(u/2)^r}{r!}\right]^2$

17

- β controls the minimum stopband attenuation of the windowed filter response
- β is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth

Note: it provides no control over passband ripples

FIR Filter Design Example

- Specifications: $\omega_p = 0.3\pi$, $\omega_s = 0.5\pi$, $\alpha_s = 40$ dB
- Thus $\omega_c = (\omega_p + \omega_s)/2 = 0.4\pi$

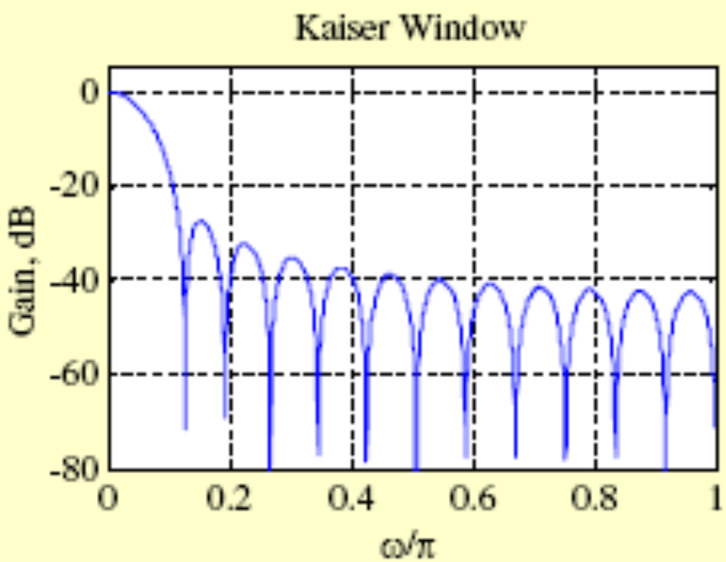
$$\delta_s = 10^{-\alpha_s/20} = 0.01$$

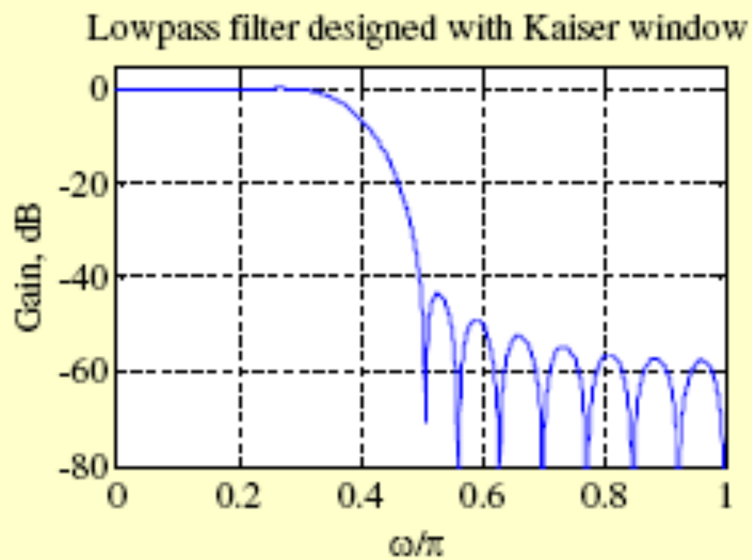
$$\beta = 0.5842(19)^{0.4} + 0.07886 \times 19 = 3.3953$$

$$N = \frac{32}{2.285(0.2\pi)} = 22.2886$$

- Choose $N = 24$ implying $M = 12$

- Hence $h_t[n] = \frac{\sin(0.4\pi n)}{\pi n} \cdot w[n], \quad -12 \le n \le 12$
  where $w[n]$ is the n-th coefficient of a length-25 Kaiser window with $\beta = 3.3953$





Summary of the window method of calculating FIR filter coefficients

- *Step 1* Specify the 'ideal' or desired frequency response of filter, $H_D(\omega)$.
- *Step 2* Obtain the impulse response, $h_D(n)$, of the desired filter by evaluating the inverse Fourier transform (Equation 7.6b). For the standard frequency selective filters the expressions for $h_D(n)$ are summarized in Table 7.2.

- *Step 3* Select a window function that satisfies the passband or attenuation specifications and then determine the number of filter coefficients using the appropriate relationship between the filter length and the transition width, Δf (expressed as a fraction of the sampling frequency).
- *Step 4* Obtain values of $w(n)$ for the chosen window function and the values of the actual FIR coefficients, $h(n)$, by multiplying $h_D(n)$ by $w(n)$:

$$h(n) = h_D(n)w(n) \tag{7.12}$$

It is clear that the window method is straightforward and involves a minimal amount of computational effort. Indeed you could obtain the coefficients with your pocket calculators. However, a PC-based program is available on the CD in the companion handbook (see the Preface for details) for calculating $h(n)$. It should be said that the resulting filter is not optimal, that is in many cases a filter with a smaller number of coefficients can be designed using other methods.